Teaching Year 6 with Numicon from September 2015

This guidance has been compiled to support teachers in delivering the Year 6 Programme of Study for England from September 2015 up until the publication of *Numicon Number, Pattern and Calculating 6* and *Geometry, Measurement and Statistics 6* in spring 2016. For more information about these forthcoming materials and the Numicon 5 materials mentioned below, go to www.oxfordprimary.co.uk/numicon.

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Contents

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Key new objectives in 2014 Year 6 Programme of Study

The tables below highlight some of the most significant differences between the 2014 Programme of Study (PoS) for Year 6 and the 2006 Primary Mathematics Framework. The lists are by no means exhaustive but may be a useful reminder of key content areas that have been changed.

Key new objectives in Y6 PoS
Number – fractions (including decimals and percentages)
Add and subtract fractions with different denominators and mixed numbers, using the concept of equivalent fractions
Multiply simple pairs of proper fractions, writing the answer in its simplest form [for example, $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$]
Divide proper fractions by whole numbers [for example, $\frac{1}{3} \div 2 = \frac{1}{6}$]

Ratio and proportion
Solve problems involving similar shapes where the scale factor is known or can be found

Geometry – properties of shapes
Compare and classify geometric shapes based on their properties and sizes and find unknown angles in any triangles, quadrilaterals, and regular polygons
Illustrate and name parts of circles, including radius, diameter and circumference and know that the diameter is twice the radius

Measurement
Convert between miles and kilometres
Recognise that shapes with the same areas can have different perimeters and vice versa
Recognise when it is possible to use formulae for area and volume of shapes
Calculate the area of parallelograms and triangles
Calculate, estimate and compare volume of cubes and cuboids using standard units, including cubic centimetres (cm^3) and cubic metres (m^3), and extending to other units [for example, mm^3 and km^3].

Algebra
Use simple formulae
Generate and describe linear number sequences
Express missing number problems algebraically
Find pairs of numbers that satisfy an equation with two unknowns
Enumerate possibilities of combinations of two variables.

Y6 topics from Primary Mathematics Framework 2006 moved elsewhere in progression
Moved to lower in Key Stage 2
Number and place value Recognizing and using thousandths and relating them to tenths, hundredths and decimal equivalents moves to Year 5
Multiplication facts are expected lower in KS2
Prime numbers as new learning starts in Y5
Number – addition, subtraction, multiplication and division Finding percentages and fractions of whole numbers is expected in Y5
Using coordinates in the first quadrant to draw, locate and complete shapes moves to Y4
Metric/imperial conversion as new learning starts in Y5
Calculating the perimeter and area of rectilinear shapes, and estimating the area of an irregular shape by counting squares moves to Y4/5

Moved to Key Stage 3
Calculator skills move to KS3 PoS
Rotation of simple shapes on the coordinate plane moves to KS3 PoS
Probability moves to KS3 PoS
Finding the mode, median and range of a set of data moves to KS3 PoS

Making use of Numicon 5 for Year 6 topics

The columns of the table below show how the curriculum content for Year 6 builds on similar objectives in Year 5, with links to the most relevant Activity Groups from *Numicon Number, Pattern and Calculating 5* and *Geometry, Measurement and Statistics 5*. There are also specific suggestions for how a Year 6 teacher might develop activities in the Numicon 5 materials to move into the Year 6 curriculum content.

In the last column, activities from the *Number, Pattern and Calculating 5 Teaching Resource Handbook* are shown with stripes of red (Pattern and Algebra), yellow (Numbers and the Number System) and dark blue (Calculating). Activities from the *Geometry, Measurement and Statistics 5 Teaching Resource Handbook* are shown with stripes of green (Geometry) and purple (Measurement).

Year 5 objective	Year 6 objective	Key additional content for Year 6	Links to related work in Numicon 5 & notes
Number – number and place value			
Read, write, order and compare numbers to at least 1 000 000 and determine the value of each digit	Read, write, order and compare numbers up to 10 000 000 and determine the value of each digit	In Year 6, the range of numbers is increased from 1 million to 10 million	Numbers and the Number System 1 – extend number range past 1 million when children are ready
Interpret negative numbers in context, count forwards and backwards with positive and negative whole numbers, including through zero	Use negative numbers in context, and calculate intervals across zero	Children calculate intervals rather than counting through zero; they use negative numbers, rather than just interpreting them	Numbers and the Number System 5
Round any number up to 1 000 000 to the nearest 10, 100, 1000, 10 000 and 100 000	Round any whole number to a required degree of accuracy	This builds on the Y5 objective: it implies children will work with increased range and accuracy beyond powers of 10 (e.g. nearest 5, etc)	Numbers and the Number System 4 – extend to different degrees of accuracy other than powers of 10
Solve number problems and practical problems that involve all of the above	Solve number and practical problems that involve all of the above.	Continues the Y5 objective	Calculating 16 (problem-solving with steps)

Year 5 objective	Year 6 objective	Key additional content for Year 6	Links to related work in Numicon 5 & notes
Number – addition, subtraction, multiplication and division			
Multiply numbers up to 4 digits by a one- or two-digit number using a formal written method, including long multiplication for two-digit numbers	Multiply multi-digit numbers up to 4 digits by a two-digit whole number using the formal written method of long multiplication	Continues and strengthens learning from Y5 on long multiplication by 2-digit numbers	Calculating 8 (mental multiplying), Calculating 12 introduces long multiplying
Divide numbers up to 4 digits by a one-digit number using the formal written method of short division and interpret remainders appropriately for the context	Divide numbers up to 4 digits by a two-digit whole number using the formal written method of long division, and interpret remainders as whole number remainders, fractions, or by rounding, as appropriate for the context	Moves on from short to long division and starts dividing by 2-digit numbers. Objective gives more detail on how to interpret remainders.	Calculating 8-9 (mental division/dividing with remainders), Calculating 13 (long dividing)
	Divide numbers up to 4 digits by a two-digit number using the formal written method of short division where appropriate, interpreting remainders according to the context	This continues the Y5 short division objective	Calculating 9 (dividing with remainders)
Add and subtract numbers mentally with increasingly large numbers	Perform mental calculations, including with mixed operations and large numbers	This combines two Y5 objectives and expands to multi-step problems (mixing operations)	Calculating 16 (problem-solving with steps)
Multiply and divide numbers mentally drawing upon known facts			

Year 5 objective	Year 6 objective	Key additional content for Year 6	Links to related work in Numicon 5 & notes
Number – addition, subtraction, multiplication and division			
Identify multiples and factors, including finding all factor pairs of a number, and common factors of two numbers	Identify common factors, common multiples and prime numbers	The topic is expanded to common multiples as well as factors	Pattern and Algebra 3 (multiples and factors)
Know and use the vocabulary of prime numbers, prime factors and composite (non-prime) numbers		The topic of prime numbers is combined with work on multiples and factors	Pattern and Algebra 3, especially Activities 6 and 7 (prime numbers)
Establish whether a number up to 100 is prime and recall prime numbers up to 19			
	Use their knowledge of the order of operations to carry out calculations involving the four operations	New learning	Pattern and Algebra 5 Activity 6 introduces brackets to clarify order of operations
Solve addition and subtraction multi-step problems in contexts, deciding which operations and methods to use and why.	Solve addition and subtraction multi-step problems in contexts, deciding which operations and methods to use and why	This continues the Y5 objective	Calculating 16 (problem-solving with steps)
Add and subtract whole numbers with more than 4 digits, including using formal written methods (columnar addition and subtraction)	Solve problems involving addition, subtraction, multiplication and division	This objective expands problem solving to all 4 operations: it is more general than similar Y5 objectives, with nothing specific about factors, multiples/ fractions/squares, cubes	Calculating 16 (problem-solving with steps)
Multiply and divide whole numbers and those involving decimals by 10, 100 and 1000			Calculating 7
Solve problems involving multiplication and division including using their knowledge of factors and multiples, squares and cubes			
Solve problems involving addition, subtraction, multiplication and division and a combination of these, including understanding the meaning of the equals sign			
Solve problems involving multiplication and division, including scaling by simple fractions and problems involving simple rates.			
Use rounding to check answers to calculations and determine, in the context of a problem, levels of accuracy	Use estimation to check answers to calculations and determine, in the context of a problem, an appropriate degree of accuracy.	This is similar to the Y5 objective	Numbers and the Number System 4 (estimating).
			Estimating before calculating is covered in Calculating 5 and Calculating 6 (written adding and subtracting)

Year 5 objective	Year 6 objective	Key additional content for Year 6	Links to related work in Numicon 5 & notes
Number – fractions (including decimals and percentages)			
Use common factors to simplify fractions; use common multiples to express fractions in the same denomination	Use common factors to simplify fractions; use common multiples to express fractions in the same denomination	This continues the Y5 objective	Numbers and the Number System 6
Compare and order fractions whose denominators are all multiples of the same number	Compare and order fractions, including fractions > 1	The objective extends comparing and ordering beyond multiples of the same denominator to any fraction, as well as top-heavy fractions or mixed numbers	Numbers and the Number System 6 Activity 2 (comparing fractions with same denominator) – extend range of numbers used in activities
Recognise mixed numbers and improper fractions and convert from one form to the other and write mathematical statements > 1 as a mixed number [for example, $\frac{2}{5} + \frac{4}{5} = \frac{6}{5} = 1\frac{1}{5}$]			Calculating 15 (& Calculating 14 Activity 5)
Identify, name and write equivalent fractions of a given fraction, represented visually, including tenths and hundredths	Add and subtract fractions with different denominators and mixed numbers, using the concept of equivalent fractions	In Y5 children work with fractions where the denominators are multiples of the same number In Y6 they may have to convert fractions to get a common denominator	Calculating 15 – denominators are multiples of the same number. Numbers and the Number System 2 begins equivalent fractions/ comparing, developed in Numbers and the Number System 6 where they start to compare fractions with a common denominator (and have to find the common denominator.)
Add and subtract fractions with the same denominator and denominators that are multiples of the same number			Numbers and the Number System 6, Calculating 15
Multiply proper fractions and mixed numbers by whole numbers, supported by materials and diagrams	Multiply simple pairs of proper fractions, writing the answer in its simplest form [for example, $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$]	Extends the Y5 objective to multiplying a fraction by a fraction	Calculating 15 covers fraction × whole number: extend this to fraction × fraction
	Divide proper fractions by whole numbers [for example, $\frac{1}{3} \div 2 = \frac{1}{6}$]	New learning	Calculating 15 Activity 5 (multiply a fraction by a whole number) – this could be used as a starting point for dividing a fraction by a whole number.
	Associate a fraction with division and calculate decimal fraction equivalents [for example, 0.375] for a simple fraction [for example, $\frac{3}{8}$]	New learning	Calculating 9 Activities 6 and 7
Recognise and use thousandths and relate them to tenths, hundredths and decimal equivalents	Identify the value of each digit in numbers given to three decimal places and multiply and divide numbers by 10, 100 and 1000 giving answers up to three decimal places	This is similar to Y5 objective	Calculating 13 Activity 4
Multiply numbers up to 4 digits by a one- or two-digit number using a formal written method, including long multiplication for two-digit numbers	Multiply one-digit numbers with up to two decimal places by whole numbers	Y6 objective extends multiplying methods to multiplying decimals: children can use written methods already learned, paying attention to place value	Calculating 12 (written methods of multiplying) together with Calculating 7 (multiply and divide by 10, 100 and 1000) for use of place value

Year 5 objective	Year 6 objective	Key additional content for Year 6	Links to related work in Numicon 5 & notes
Number – fractions (including decimals and percentages)			
Divide numbers up to 4 digits by a one-digit number using the formal written method of short division and interpret remainders appropriately for the context	Use written division methods in cases where the answer has up to two decimal places	Y6 objective specifies more accurate division answers, using decimal places rather than remainders	Calculating 13 (with Calculating 7 for reference to place value)
Round decimals with two decimal places to the nearest whole number and to one decimal place	Solve problems which require answers to be rounded to specified degrees of accuracy	In Y6 the objective expands to finding any degree of accuracy, not just the nearest whole or 1 decimal place	Numbers and the Number System 4 (especially Activities 7 and 8)
Read and write decimal numbers as fractions [for example, $0.71 = \frac{71}{100}$]	Recall and use equivalences between simple fractions, decimals and percentages, including in different contexts.	Y6 objective extends use of equivalence	Calculating 11 especially Activity 6, and Numbers and the Number System 7 (fraction/decimal/percentage conversion)
Recognise the per cent symbol (%) and understand that per cent relates to 'number of parts per hundred', and write percentages as a fraction with denominator 100, and as a decimal			
Solve problems which require knowing percentage and decimal equivalents of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{4}{5}$ and those fractions with a denominator of a multiple of 10 or 25.			

Year 5 objective	Year 6 objective	Key additional content for Year 6	Links to related work in Numicon 5 & notes
Ratio and proportion			
Pupils use multiplication and division as inverses to support the introduction of ratio in year 6, for example, by multiplying and dividing by powers of 10 in scale drawings or by multiplying and dividing by powers of a 1000 in converting between units such as kilometres and metres. **(non-statutory guidance, Number – multiplication and division)**	Solve problems involving the relative sizes of two quantities where missing values can be found by using integer multiplication and division facts	All work in Y5 was in non-statutory sections, so this is formalised for Y6	Pattern and Algebra 2 covers using inverse to solve missing number or working backwards through problems: could use this as a basis to continue finding missing numbers
Pupils should be taught throughout that percentages, decimals and fractions are different ways of expressing proportions. **(non-statutory, Number – fractions including decimals and percentage)**	Solve problems involving the calculation of percentages [for example, of measures, and such as 15% of 360] and the use of percentages for comparison		Calculating 11, Numbers and the Number System 7
Pupils calculate the area from scale drawings using given measurements. **(non-statutory guidance, Measurement)**	Solve problems involving similar shapes where the scale factor is known or can be found		Measurement 6
Solve problems involving multiplication and division including using their knowledge of factors and multiples, squares and cubes	Solve problems involving unequal sharing and grouping using knowledge of fractions and multiples.	In Y6, problem-solving expands to include unequal sharing and grouping as well as equal dividing and multiplying	Calculating 13 Act 1 covers sharing/grouping for dividing. Pattern and Algebra 3 covers factors/multiples, Pattern and Algebra 4 (Activity 1) moves on to general statements with divisibility. Calculating 10 looks at proportion and ratio – sharing in a ratio is an example of unequal sharing.

Year 5 objective	Year 6 objective	Key additional content for Year 6	Links to related work in Numicon 5 & notes
	Algebra	NEW SECTION FOR YEAR 6: all new learning	
	Use simple formulae		Measurement 5: simple formulae for the area and perimeter of rectangle
	Generate and describe linear number sequences		Pattern and Algebra 1 and Pattern and Algebra 4
	Express missing number problems algebraically		Pattern and Algebra 5: Activity 2 looks at missing number inequalities, Activity 5 starts using symbols for unknowns
	Find pairs of numbers that satisfy an equation with two unknowns		Pattern and Algebra 1 covers sequences
	Enumerate possibilities of combinations of two variables.		Measurement 5 – using formulae for rectangles. Children could look at different options for n and b based on a certain area, for example.

Measurement			
Convert between different units of metric measure (for example, kilometre and metre; centimetre and metre; centimetre and millimetre; gram and kilogram; litre and millilitre)	Solve problems involving the calculation and conversion of units of measure, using decimal notation up to three decimal places where appropriate	Extends to decimals with conversion	Measurement 1 – children could increase level of accuracy of conversion to take it to 3 decimal places
Use all four operations to solve problems involving measure [for example, length, mass, volume, money] using decimal notation, including scaling.	Use, read, write and convert between standard units, converting measurements of length, mass, volume and time from a smaller unit of measure to a larger unit, and vice versa, using decimal notation to up to three decimal places	Decimals specified up to 3 decimal places	Measurement 1 (metric/ imperial), Measurement 7 (problem-solving with units of time, money etc, including converting)
Understand and use approximate equivalences between metric units and common imperial units such as inches, pounds and pints	Convert between miles and kilometres	Converting between metric and imperial is extended to mile/km	Measurement 1
Measure and calculate the perimeter of composite rectilinear shapes in centimetres and metres	Recognise that shapes with the same areas can have different perimeters and vice versa	New learning	Measurement 3, Measurement 5
	Recognise when it is possible to use formulae for area and volume of shapes	New learning	Measurement 3, Measurement 5
Calculate and compare the area of rectangles (including squares), and including using standard units, square centimetres (cm^2) and square metres (m^2) and estimate the area of irregular shapes	Calculate the area of parallelograms and triangles	New learning	Measurement 5 covers area of rectilinear shapes, as a starting point for other areas
Estimate volume [for example, using 1 cm^3 blocks to build cuboids (including cubes)] and capacity [for example, using water]	Calculate, estimate and compare volume of cubes and cuboids using standard units, including cubic centimetres (cm^3)	In Year 6, children compare and calculate volume as well as estimating (in Y5 they only estimate).	Measurement 4

Year 5 objective	Year 6 objective	Key additional content for Year 6	Links to related work in Numicon 5 & notes
Geometry – properties of shapes			
	Draw 2-D shapes using given dimensions and angles	New learning	Geometry 1, Geometry 3
Identify 3-D shapes, including cubes and other cuboids, from 2-D representations	Recognise, describe and build simple 3-D shapes, including making nets	The objective extends to building 3D shapes as well as identifying them.	Measurement 4 covers building cubes: children could extend this to cuboids and from there look at nets.
Use the properties of rectangles to deduce related facts and find missing lengths and angles Distinguish between regular and irregular polygons based on reasoning about equal sides and angles.	Compare and classify geometric shapes based on their properties and sizes and find unknown angles in any triangles, quadrilaterals, and regular polygons	The Y6 objective extends beyond rectangles (Y5) to other types of polygon.	Geometry 3, Measurement 5 – triangles and quadrilaterals only
	Illustrate and name parts of circles, including radius, diameter and circumference and know that the diameter is twice the radius	New learning	(not covered in Y5)
Draw given angles, and measure them in degrees (°) Know angles are measured in degrees: estimate and compare acute, obtuse and reflex angles Identify: • Angles at a point and one whole turn (total 360°), • Angles at a point on a straight line and $\frac{1}{2}$ a turn (total 180°), • Other multiples of 90°	Recognise angles where they meet at a point, are on a straight line, or are vertically opposite, and find missing angles.	Y6 objective uses Y5 learning about angles on line and at a point.	Geometry 1, Activity 5 – angles at a point and on a line

Year 5 objective	Year 6 objective	Key additional content for Year 6	Links to related work in Numicon 5 & notes
Geometry – position and direction			
Identify, describe and represent the position of a shape following a reflection or translation, using the appropriate language, and know that the shape has not changed.	Describe positions on the full coordinate grid (all four quadrants)	New learning	Measurement 2 uses the first quadrant – could extend this to negative number data
(non-statutory) Pupils recognise and use reflection and translation in a variety of diagrams, including continuing to use a 2-D grid and coordinates in the first quadrant. Reflection should be in lines that are parallel to the axes.	Draw and translate simple shapes on the coordinate plane, and reflect them in the axes.	In Y5 they translate shapes, but in Y6 they do it in relation to the 4 quadrants of the full coordinate grid and reflect shapes in the axes.	Geometry 2 (reflect/ translate on coordinate grid): extend to the other quadrants of the grid

Year 5 objective	Year 6 objective	Key additional content for Year 6	Links to related work in Numicon 5 & notes
Statistics			
Solve comparison, sum and difference problems using information presented in a line graph	Interpret and construct pie charts and line graphs and use these to solve problems	Pie charts are new learning	Measurement 2, Activity 6 covers pie charts and bar charts. Measurement 7 Activity 3 uses a line graph
Complete, read and interpret information in tables, including timetables.	Calculate and interpret the mean as an average.	The mean/average is new learning	(not covered in Y5)

Key mathematical ideas for Number, Pattern and Calculating 6 (selected pages)

Underlying the activities in Number, Pattern and Calculating 6 are many key mathematical ideas that children will be developing and extending, as well as some symbolic conventions they may be meeting for the first time.

In order to teach these ideas and conventions effectively, those who are working on activities with children will need to be very clear themselves about the mathematical content involved in each activity.

The following section includes a brief outline of the key mathematical ideas that children will be encountering in the activity groups of the *Number, Pattern and Calculating 6 Teaching Resource Handbook*. Discussion of the mathematical ideas within this section will help with planning how best to develop these ideas with children.

The educational context page of each activity group in the Teaching Resource Handbooks lists the key mathematical ideas associated with each activity. As you prepare for your teaching, you may find it helpful to remind yourself about the key ideas behind each activity by referring to the following section.

The maths coordinator may also find it useful to work on these key mathematical ideas in professional development sessions with the class teachers and the wider school staff.

Click on any of the items in the contents to go directly to that section.

In this section you will find overviews of:

Pattern and algebra: essential to mathematics for children of all ages

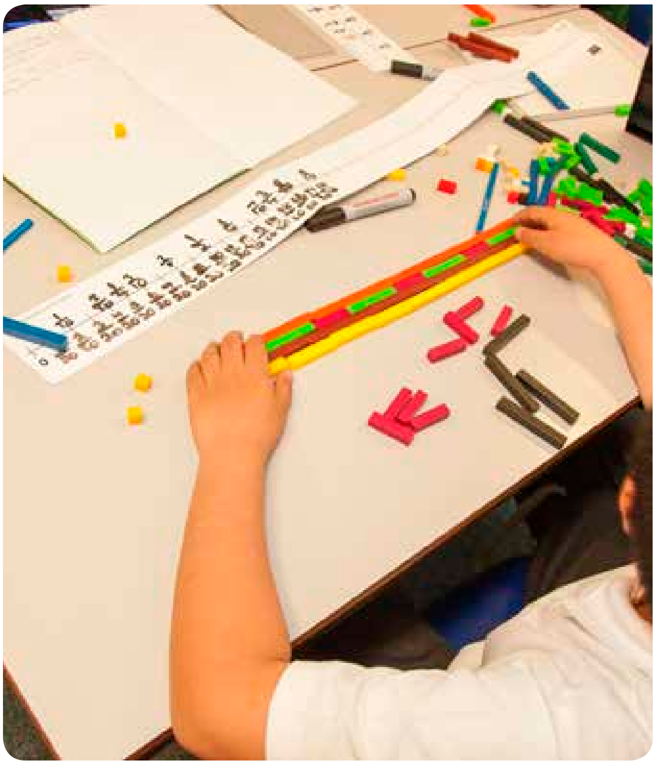
An essential idea underlying all Numicon activities is that of pattern. Pattern may not sound like a particularly mathematical idea, as we are used to patterns of one sort or another occurring in so many non-mathematical contexts. We could not have learned to speak, for instance, without noticing patterns in the sounds we heard as infants; patterns (rhythms) structure music and dance, and patterns in stories and plays enable us to anticipate the unfolding of a plot (they also, of course, allow our expectations to be manipulated by writers and composers). Much poetry depends upon patterns for its effect, and most scientific research is an attempt to discover or establish patterns in observable phenomena.

Importantly, it is the detection of patterns in our experiences that makes essential aspects of our lives predictable. And since being able to predict successfully is absolutely vital to human survival, seeing patterns is also something humans generally do well. Seeing patterns is what enables us to generalize and then to predict what comes next, thus gaining a degree of control over our environment and our futures.

Patterns are essential in mathematics for a very special reason: they enable us to imagine actions, events and sequences going on 'forever' without us having physically to work out and wait for each and every step. It is patterns that allow us to generalize into the future. Counting is a good example. As we have invented a system for generating number names, we can imagine what it would be to count forever without ever actually having to do it. Most people know they could count to one million, without ever having done it. They know they could because they know the place value system; they know the patterns in number names that would enable them to go on forever. Importantly, once children see the pattern that each next whole number is 'one more' than the previous one, they also know how counting things may go on forever – a vital generalization that allows children to work with collections of any size.

As another example, by noticing the pattern that it doesn't matter which way round you multiply any two numbers (a generalization we call the **commutative property**), children gain the insight that they only have to remember half of their multiplication tables. And because they have generalized in this way, they don't have to keep checking every example.

It is impossible to overestimate the importance of pattern to mathematical thinking. In fact, a very large part of algebra, often thought of as the most powerful branch of mathematics, consists of seeing, manipulating and generalizing from patterns. It is important to remember that in all the key mathematical ideas discussed here, pattern and generalization are fundamental.



Formal algebra

For many people, doing 'algebra' means 'using letters instead of numbers', and there is some rough sense in this. There are occasions when we want to talk about either a range of numbers (and not one particular number), or about a particular value when we don't know what it is. In both situations, since (for these two quite different reasons) we can't specify *particular* numbers, we use letters to talk about relationships between numbers instead.

Thus sometimes we use letters to stand for *unknown* numbers and sometimes we use letters to stand for a known *range* of numbers. These two very different uses are both technically described as using letters as 'variables' because the value(s) a letter is used to stand for can 'vary'.

Suppose we had 48 square floor tiles to tile along a corridor, that each tile is 40 cm square, and that the width of the corridor is 1 m 20 cm. How far would these tiles stretch along the corridor? We could reason that three tiles will fit nicely across the width, and that 'n' (the number of tiles that would stretch along the corridor) would therefore be related to the other numbers and measurements by the following equation:

$$n = 48 \div 3$$

And then we could say that the actual distance in centimetres ('L') that the tiles would stretch along the corridor would be given by:

$$L = n \times 40\,\text{cm}$$

In this situation we can use the letters 'n' and 'L' to 'hold' our reasoning through the problem; here we use these letters to represent the unknown numbers that we are trying to work out as we set up our plan for working out their values. Encouraging children to express their reasoning using letters to stand for unknown amounts within a problem situation is very important at this stage of their progress; it enables children to express simply and clearly the relationships (and thus the reasoning) they see as crucial to the problem solution. That is using letters as 'unknowns'.

In the Number, Pattern and Calculating 6 Teaching Resource Handbook (Pattern and Algebra 3), children progress from finding unknowns in single 'empty box' number sentences such as $\square - 24 = 37$, to finding pairs of numbers that satisfy number sentences involving two unknowns. Children will need to think systematically to explore the possible pairs of numbers that will satisfy number sentences such as $\triangle + \square = 7$, or $12 - \triangle = \square$. It is important to vary the operations involved as well, and to explore number pairs that satisfy sentences such as $\square \times 8 = \triangle$, or even $a \div b = 12$.

When we need to generalize about a relationship between numbers (or measures) we use letters because we want to stress that we are talking about whole sets of numbers, not just particular numbers. So when we want to express our generalization that 'it doesn't matter which way around you add two numbers, their total will be the same', we can write the following 'identity':

$$a + b = b + a$$

Similarly, when we want to express the **generalization** that relates the area of a rectangle to the lengths of its sides, we write:

$$A = l \times b$$

because we are stating that this relationship works for *any* values of 'A' and 'l' and 'b'. This is using letters as 'variables'; 'A' and 'l' and 'b' each stand for a range of numbers.

We use formulae to calculate unknown amounts; if we know the lengths of the sides of a rectangle and want to work out its area, then in this specific context the letter 'A' represents the unknown amount, and we calculate it by feeding the particular values for 'l' and 'b' into the formula. Thus depending upon any given context, using formulae can invite children to use letters to stand for both 'unknowns' and for generalizing over ranges of values.

It is important to recognize that the formula $A = l \times b$ is also an example of a **function** (see below). We could say that 'the area of a rectangle (A) is a function of the lengths of its sides (l and b)', by which we mean that the area of a rectangle *depends upon* the lengths of its sides.

Another kind of situation in which letters used as variables become increasingly important lies in describing the rules that lie behind regular sequences of numbers, e.g. the sequence 2, 4, 6, 8, 10 … (see Pattern and Algebra 2). We could describe the rule behind this simple sequence in words by saying something like, 'It starts at 2, and then moves on in steps of 2,

for as long as we like.' Or we could say, 'It's the sequence of even numbers, beginning with 2.' These descriptions of the sequence tend to focus on how we move from one term to the next, having started at 2, and are sometimes called 'sequential generalizations' or the 'term-to-term rule'.

But another way of describing this regular sequence involves a different kind of generalization, sometimes called a 'global generalization'. In this approach we describe the sequence by constructing what we call 'the general term' by using letters (usually 'n') to describe *any* term in the sequence in terms of its *position* in the sequence. So in this sequence we could notice that the first term is '2', the second term is '4', and so on, and write this out fully as two parallel, matched sequences:

Position:	1	2	3	4	5	6	7	8 …
Term:	2	4	6	8	10	12	14	16 …

We could then notice that each term in the sequence has a numerical value exactly twice as big as its position number in the sequence, e.g. the 8^{th} term in the sequence has the value 2×8, i.e. 16. This allows us to generalize and describe 'the rule' for the sequence by saying 'every term is twice the value of its position number'.

Using the letter 'n' (as a variable) to represent any position number, we can now describe the general term for this sequence as '$2 \times n$', or '$2n$', i.e. the rule is: you can work out the value of any term in this sequence by simply doubling its position number.

In many situations, knowing the general term for a sequence is much more helpful than knowing the term-to-term rule. For instance, in this case if we want to know the 213^{th} term in the

sequence, we simply double 213 and arrive at '426' as the 213th term. This is a lot more efficient than adding 2, and 2, and 2, and 2 …, from the beginning, until we have added 2, 212 times.

Number, Pattern and Calculating 5 laid some important foundations for children's work on identifying the rules of sequences by creating several series of growing visual patterns with number rods (*Number, Pattern and Calculating 5 Teaching Resource Handbook*, Pattern and Algebra 4). These activities offered children ways of thinking visually about number sequences that will help them in determining both term-to-term rules and general terms, and such illustrating will again be useful in Number, Pattern and Calculating 6, Pattern and Algebra 2 as children learn to translate visual and verbal understanding of number sequences into symbolic expressions.

An increasingly important aspect of work in Number, Pattern and Calculating 6 involves children learning to use letters to express relationships between numbers and unknown quantities of various kinds symbolically. This aspect of children's mathematical communicating – and hence their mathematical thinking – develops significantly at this stage, particularly in relation to the idea of a **Function**.

Functions – a special kind of pattern in mathematics

In Number, Pattern and Calculating 6 children continue to work on particular patterns that are called 'functions'. Mathematical functions are used to handle relationships of dependence between changing values, for example, the relationship between time, speed, and distance travelled as we move. The distance travelled at any point depends upon how fast we have been going and how long we have been travelling.

Central to the general idea of a function is that of a 'variable'; this is what mathematicians call whatever it is that is changing. In the travelling example, the variables we speak about are time, speed and distance travelled, all of which change in relation to each other. In Number, Pattern and Calculating 6, children begin to learn to handle functions symbolically with formulae; in the travelling example the formula expressing the function would be $s = \frac{d}{t}$, which could be read as, 'Speed is equivalent to distance travelled divided by time taken.' If you travelled 180 kilometres (d) in 3 hours (t), your average speed (s) was 180 ÷ 3, which equals 60 kilometres per hour. The quantities d, t, and s are said to be 'variables' because they vary, but their interrelationship – the 'function' relationship expressed in the formula, or how they depend on each other – remains the same. Functions are, importantly, another example of generalizations – or seeing the general patterns of relationships.

In our discussion of **Formal algebra** (above), we speak about how formulae are used to express relationships of dependence between variables symbolically, both in the example of $A = l \times b$ (in which the area of a rectangle depends upon the lengths of its sides), and in the example of the general term of the number sequence 2, 4, 6, 8, 10, 12 … – general term '$2n$' (in which the value of a term in the sequence (calculated as $2 \times n$) depends upon its position (n) in the sequence). We could express this 'general term' rule for the sequence symbolically with the formula $T = 2n$, in which the value of a term, i.e. 'T', depends upon the value of 'n', its position. Alternatively, we could just say in words that 'the value of a term in this sequence is a function of its position in the sequence'.

Because functions are about relationships of dependence, a distinction is sometimes drawn between 'independent' and 'dependent' variables. Sometimes the independent variable is called the 'input' to the function, and the dependent variable is called the 'output'. It is very important to remember that for a relationship to be called a function in mathematics, any given input must determine exactly one unique output (otherwise we wouldn't always know for example, how 'Area' depends upon 'lengths of sides', or indeed if it always does). If we interpret the formula $A = l \times b$ as showing how the area of a rectangle depends upon the lengths of its sides, then 'l' and 'b' are called the independent variables, and 'A' is called the dependent variable; A depends upon l and b, and for every pair of values of 'l' and 'b' there is just one associated value for 'A'.

In Number, Pattern and Calculating 4 children were invited to generalize in finding a connection between the first few terms of a sequence and the *sum* of those terms, and also to describe the general term of a regular sequence of numbers. At this stage children were encouraged to express their

generalizing in words, and use the imagery of number rods to support their describing.

In Number, Pattern and Calculating 5 children began to focus more precisely on what are called 'linear sequences'. Any 'number sequence' in mathematics is simply a collection of numbers set out in an order of some kind; 1, 3, 6, 10 ... is a number sequence. *Linear* number sequences are so-called because if we plotted a graph with them, that graph would produce a perfectly straight line; this happens only when the numbers involved have a constant difference (or step, up or down) between them.

The sequence discussed above of 2, 4, 6, 8, 10 … is a linear sequence, and if we were to plot the pairs of positions (n) and associated values (T) on a graph we would plot the points (1, 2), (2, 4), (3, 6), (4, 8) …, which all lie along a straight line. Sensibly enough, the **function** relationship that connects values of T and n together ($T = 2n$) is also called a 'linear function'.

In Number, Pattern and Calculating 5 children also considered number sequences with more complicated rules, but at that stage the crucial emphasis was still upon showing these values *visually* (most often with number rods) so that children could literally 'see' how sequences grow, and describe in words how terms grew using visual language. By subsequently assigning number values to the rods involved, the growth of sequences could be described in numerical terms.

With the work on formulae in Number, Pattern and Calculating 6, and on the general terms of sequences, children begin to express generalized relationships of dependence between varying values (**functions**), symbolically. It is always helpful to encourage children to explore relationships visually with illustrations, and to describe relationships in words, before honing their general expression down to concise, symbolic formulae. Functions are a very, very important idea in mathematics, and it is essential that children come to handle these relationships securely.

Patterns in using the four operations: important algebraic relationships

As children begin to calculate in practice, we can help them to notice particular patterns (or connections) in their use of numbers and then to generalize from what they have noticed. Usually we do this without deciding to give these features a formal mathematical name. For instance, we can help children to notice that it doesn't matter which way round we do any multiplying, we will always get the same answer, and then invite them to use that observation to calculate 4 × 7 if they can't remember 7 × 4. We may do that in practice, but we don't often formally call it 'the **commutative property** of multiplying', although we might say 4 × 7 and 7 × 4 are **equivalent**.

However, ideas about how the arithmetic operations work together are important algebraic ideas – they are about number relationships – and consequently within mathematics they do have formal names. Whether or not we think it important that children know and use these formal mathematical names, it is important for children's mastery of calculating that they understand what are called the 'properties' of, and relationships between, the four operations with numbers, especially as they become older and approach formal algebra. In what follows, we use the formal mathematical names for those properties and algebraic relationships that are important at this stage.

Equivalence

Equivalence is one of the most important mathematical relationships of all, and yet it is often the case that not enough attention is paid to it explicitly as we discuss work with children. Children often work with equivalence implicitly from very early on in their thinking, but in doing mathematics at this stage we definitely need to discuss instances of this relationship fully and explicitly, and allow children plenty of time to reflect.

We signal an equivalence relationship in mathematics by using the symbol '=', for example by writing:

$$\frac{4}{5} = 0{\cdot}8 \text{ or } 3 + 4 = 7$$

Equivalence literally means 'equal value'. Quite often the most interesting and important instances of equivalence occur when two or more things are of equal value but look different. In early calculating, there are at least three occasions when children face important instances of equivalence: the introduction of the '=' sign itself (which

means 'is equivalent to'), when they encounter quantity value and column value (see Names for numbers: counting numbers and place value), and when they meet fractions, decimals and percentages ($\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$ = 0·5 = 50% = 1 ÷ 2 …). However, there are numerous other occasions when children need to see equivalence and we are often less explicit.

For their mental calculating strategies to make sense, children have to be able to see what are called the 'decompositions' of any number as equivalent to each other, for example 9 = 1 + 8 = 3 + 6 = 10 − 1 etc. Similarly, factor pairs are all equivalent, e.g. 1 × 16 = 2 × 8 = 4 × 4 = 8 × 2 = 16 × 1. In measuring, equivalences between units (100 cm = 1 m, 1″ = 2·54 cm) are at the heart of being able to understand and relate systems of measurement.

When young children seem not to understand something that is clear to us in calculating, there is often an equivalence we see that they do not, or vice versa. We might fully realize that 4 ÷ 5 = $\frac{4}{5}$ for example, before children have come to equate the process of dividing with fractions that are simply 'objects' to them.

There is often also a language problem. As it may sometimes seem unhelpful to use the formal word 'equivalence' with children, we can often resort to an easier word, 'same', when talking about equivalence; we often say, 'It's the same thing.' Unfortunately, equivalence does not mean quite 'the same thing' – it means **equal value**, **different appearance**.

In Number, Pattern and Calculating 6, important equivalences between expressions continue to be explored. For example, as children's abilities to calculate with a wider and wider range of numbers develops, success depends upon seeing equivalences between common fractions and decimal fractions, between equivalent proper fractions, between mixed numbers and improper fractions, and between all of these and percentages. It is increasingly important to children's developing understanding and to the fluency of their calculating that they realize '0·5' and '$\frac{1}{2}$' and '50%' and '1 ÷ 2' all look different, and yet they are equivalent ways of referring to the same value.

Equivalence is the key relationship underlying efforts to 'simplify' fractions in Number, Pattern and Calculating 6, for example being able to simplify $\frac{12}{20}$ to its simplest form of $\frac{3}{5}$ (see also the section on **Fractions**). Later on in their schooling children will depend upon this relationship equally crucially as they are asked to 'simplify' increasingly complex algebraic expressions.

Inverse relationships

Adding and subtracting have what is called an 'inverse relation' to each other. What this means is that each can 'undo' the other. If I add 6 to a number, I can then undo that adding by subtracting 6, and vice versa. This knowledge is important to children for several reasons. Firstly, the more connections children can make between things they learn, the more meaningful their learning is. Secondly, it is important children don't think adding and subtracting are completely unconnected, because if they do they will never understand the 'inverse of adding' structure of subtracting. Finally, children should understand that adding or subtracting can always be checked by doing the inverse calculation: we can always check our adding by subtracting, and vice versa.

Multiplying and dividing also have an inverse relation to each other. Noticing how dividing undoes multiplying (and vice versa) is crucial to connecting these two operations with each other. This will also help children to see that, if multiplying is seen as repeated adding, it makes sense that dividing can be seen as repeated subtracting (quotition). These are important foundations for the extended multiplying and dividing calculations children develop in Number, Pattern and Calculating 6.

As noted below in relation to counting and place value, 'partitioning' numbers in various ways while calculating is the inverse action to the 'grouping objects into tens' actions that children have practised regularly in answering 'how many?' questions without counting.

Inverse relationships are also used in relation to the 'empty box' notation that was first introduced in the *Number, Pattern and Calculating Teaching Resource Handbook 1*. Children begin to learn the important algebraic use of symbols to stand for unknown amounts (see above) by using empty boxes. For instance, in asking them to solve 3 + □ = 10, we ask them to work out which number should go in the box to make the number sentence true. This type of apparent

'adding problem' requires children either to 'undo' a number fact they can remember, or to subtract 3 from 10; in either case they are using an inverse relationship. Empty box problems also ask of children a clear understanding of **equivalence**.

Although we do not yet make this explicit to children, from the *Number, Pattern and Calculating 4 Teaching Resource Handbook* onwards they have also met inverses of another kind. In negative numbers, children meet what are called the 'additive inverses' of positive numbers. Put simply, $^{-}2$ is the additive inverse of $^{+}2$, $^{-}3$ is the additive inverse of $^{+}3$ and so on. If we add any two 'additive inverses' together, the result will always be 0 (zero); in an important sense an additive inverse undoes what its partner can do – and together they are equivalent to 'doing nothing'.

There is an increased use of inverse relationships in Number, Pattern and Calculating 6 as children increasingly depend upon relating factors and multiples to each other, both in 'simplifying' and in developing their calculating with fractions.

In unit fractions, children are also meeting the 'multiplicative inverses' of whole numbers, e.g. $\frac{1}{2}$ is the multiplicative inverse of 2, $\frac{1}{3}$ is the multiplicative inverse of 3 and so on. If we multiply any pair of multiplicative inverses together, the result will always be 1. Any multiplicative inverse undoes what its partner can do: together they are equivalent to 'doing nothing' when multiplying, that is, to multiplying by 1.

We don't expect children working through Number, Pattern and Calculating 6 activities to be discussing multiplicative or additive inverses with you or with each other. What is important is having an awareness of how children are gradually learning more and more about the individual roles of numbers and operations that go together to make up what is called our 'real number system'. It is crucial that children increasingly understand how all the individual pieces – new operations, new kinds of numbers – fit together into this coherent system, and inverse relationships are an important part of that.

Zero and one: examples of 'doing nothing'

Most children notice that there's something funny about zero. Quite rightly, too: there is. Within adding and subtracting, zero is what is called an 'identity element', which means that operating with it leaves everything exactly as it was; adding or taking away zero amounts in effect to 'doing nothing'. Children need plenty of help understanding this because (again, quite rightly) they can't see the point of doing nothing. There is no point, it is simply that zero is a number – it has its own important position on the number line and it can be added and subtracted. It just gives a strange result when added or subtracted: no change at all. When multiplying or dividing, 1 is the identity element; multiplying or dividing by 1 leaves everything just as it was.

Importantly, as multiplying and dividing are introduced the role of zero becomes even more bizarre. In fact, in these operations zero becomes a kind of rogue element, destroying everything it touches. Children will find that multiplying anything by zero always results in zero itself – a very strange result. Even stranger is the fact that dividing by zero is simply not defined in mathematics; it is a calculation with no answer at all. Not many children ask about dividing by zero, although by the time they are working on Number, Pattern and Calculating 6 some may begin to do so; if you are asked, try talking through and exploring with children, 'What would happen if we tried?'

Commutative property

Adding has what is called a 'commutative property'; subtracting does not. It does not matter which way round you do an adding sum; it does matter when you are subtracting. 12 + 6 equals 6 + 12, but 12 – 6 does not equal 6 – 12. Similarly, multiplying (because of its repeated adding structure) is commutative; dividing (because of its repeated subtracting structure) is not.

Associative property

If you have three numbers to add together, it doesn't matter which pair you add first before then adding the third. With 2 + 3 + 5 for example, you can add the 2 and the 3 first, or the 2 and the 5 first, or the 3 and the 5 first. Whatever you do, you always get the same answer: 10. Because of this, adding is said to have an 'associative property'. The same applies to multiplying: try 2 × 3 × 5.

With subtracting this doesn't work. Try it out with 12 – 4 – 1. Is the answer 7 or 9? It could be either; we don't know. Examples like this explain why it is often clearer to use brackets in many numerical and algebraic expressions (and see notes on the **BODMAS** convention, below); in this case, using brackets makes the expression clear and unambiguous: (12 – 4) – 1 = 7 and 12 – (4 – 1) = 9.

Similarly, division does not have an associative property. Try 24 ÷ 2 ÷ 3. Is the answer 4 or 36? Quite a difference! Try (24 ÷ 2) ÷ 3 and compare the answer with 24 ÷ (2 ÷ 3). Once again, following **BODMAS** is necessary to make the original expression clear and unambiguous; if we have several operations of the same kind to work out, we do them in order *from left to right.*

In Number, Pattern and Calculating 4, children were introduced to the use of brackets in order to be clear about expressions such as 6 × 5 – 2 × 5, which we conventionally record as (6 × 5) – (2 × 5) so that children do not work out a value for the whole expression by carrying out the series of operations in the order they read them, from left to right. From Number, Pattern and Calculating 4 onwards, children will have met the first part of the **BODMAS** convention in learning to work out the value of whatever is inside brackets first.

Distributive property

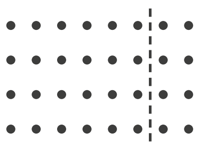
In Number, Pattern and Calculating 6, children continue to make use of a very important relationship between adding and multiplying that helps us to multiply large numbers together quite easily. To give it its full name, 'the distributive property of multiplication over addition' is what allows us to break down a large multiplying calculation into a series of smaller calculations that are easy to do mentally – provided we can recall our tables.

Expressed algebraically, this property could be summed up as:

$$a\,(b + c) = ab + ac$$

which really means, 'You can do any multiplication a bit at a time, and then add up the individual results to get the final answer.'

The key image that we use to illustrate this property is that of an array. For example:



This array can be used to illustrate that 8 × 4 = 4 × 8 (**commutative property**), and also that

$$4 \times (6 + 2) = (4 \times 6) + (4 \times 2)$$

This is the property that lies behind both traditional written methods of 'long multiplication' and the 'grid method' of

multiplication, and of course applies to all numbers, however large. Thus we can break down multiplying 247 by 7 into three more manageable chunks:

$$7 \times (200 + 40 + 7) = (7 \times 200) + (7 \times 40) + (7 \times 7)$$
$$= 1400 + 280 + 49$$
$$= 1729$$

Note that the **distributive property** is also the property that allows us to divide numbers in stages, that is, it is the property that makes possible both short and long division. For example, when dividing 276 by 6, the short division method involves dividing 270 by 6, and then (after exchanging the remainder and adding 6 units) 36 by 6; the respective results of 40 and 6, when added, give the required result of 46.

Non-computational thinking

All of the patterns and relationships described so far are used in what has in recent times become known as 'non-computational thinking', and this important work continues to be developed in Number, Pattern and Calculating 6. 'Non-computational thinking' is a term that has begun to be used to describe ways of manipulating relationships between numbers *without* actually computing a numerical outcome.

Non-computational thinking is important for at least two reasons: firstly, it helps to lay a foundation for children's formal algebraic thinking; secondly, it is often extremely useful for converting an apparently complex or difficult calculation into an **equivalent** and easier one.

As children move towards formal algebra, they move towards describing relationships between both known and unknown numbers explicitly and symbolically. As an example of something children will need to be able to work with later on, think about the equation $2x^2 - 7x + 3 = 0$, in which x represents an unknown number, or possibly, more than one number.

As children work to solve this equation, that is, to work out what values 'x' might have, they will need to manipulate this combination of known *and unknown* numbers in exactly the same way that we calculate with all known numbers. You may remember that children will need to 'factorize' this equation, i.e. to work out a pair of factors that multiplied together produce the expression on the left of the equation. Such a pair of factors turns out to be $(2x - 1)$ and $(x - 3)$, so that we can then say:

$$2x^2 - 7x + 3 = (2x - 1)(x - 3) = 0$$

and then deduce that x must either be $\frac{1}{2}$ or 3 because if:

$$(2x - 1)(x - 3) = 0$$

Then *either* $(2x - 1) = 0$ or $(x - 3) = 0$ or both of them do, because the only way in which the product of two numbers can be zero is if either one, or both, of those numbers is itself zero. Hence, if $2x - 1 = 0$, then 'x' must be $\frac{1}{2}$, or if $x - 3 = 0$, then 'x' must be 3.

To do such algebraic thinking, children will need to know how numbers work with each other *in general*. This might be called thinking 'about' number relationships as opposed to thinking 'with' particular numbers (because in this case we don't actually know what all the numbers are). And so a large part of learning to do algebraic manipulation successfully involves coming to know how to manipulate numbers, whatever they are, and in particular when we *don't know* what they are. Non-computational thinking is thus an important introduction to key work on number relationships for children, which lays very important foundations for their later mathematics. The work you do with children in Number, Pattern and Calculating 6 on these aspects of calculating are absolutely crucial for their success in mathematics at secondary school.

Useful work on such non-computational thinking can also be developed as we invite children to think about changing a potentially difficult calculation into an equivalent but easier one, before they try to compute the specific answer. For example, 580 + 260 can be changed into (600 + 260) – 20, which is equivalent and (for many people) much easier to calculate mentally. Children at this stage should always be encouraged to think about any calculation they face *before they calculate*, rather than rushing in to calculate with the particular given numbers straight away. It is always possible that there is an equivalent calculation that would be much easier to carry out.

Tasks specifically requiring non-computational thinking can always be given to children simply to get them thinking in

a 'relational' (algebraic) way. For instance, give children a statement like,

$$350 + 280 = 330 + 300$$

and ask them if they can explain why it is true *without* computing either addition calculation. The thinking required is non-computational (we don't want them to do the stated sums) and is also a great deal to do with seeing **equivalence**.

It is important to recognize that successful non-computational thinking underpins the kind of flexibility and fluency in calculating that we want all children to develop as they move into secondary schooling.

Names for numbers: counting numbers and place value

Our civilization has been very clever in devising a system for generating symbolic number names which not only allows us to go on inventing new names for counting numbers 'forever', but which also allows us to tell instantly where in the series of number names any particular name will be found. When we read '273' successfully we know that it is the name of the whole number that comes immediately after 272 and a hundred before 373. This means that we don't have to remember every individual symbolic number name and its place in the order (which would be impossible anyway, since there are an infinite number of them); we just have to master the system that generates the names.

The two essential keys to generating this infinite set of names for counting numbers are that we 'group into tens', and that we use a writing code we call 'place value'.

The first of these keys is **grouping** into tens. The number we call 'ten' (in numerals, '10') is the most important number in our naming system, because, when we are counting collections, as soon as we have ten of something we call them 'one' of something else. So ten 'ones' are called one 'ten', ten 'tens' are called one 'hundred', ten 'hundreds' are called one 'thousand', and so on. In effect, in the language we use, we are always grouping things into tens (and then grouping groups) to call them one of something else.

In children's early experiences of finding how many objects there are in a collection, it was always important to help them group collections physically into tens as they worked to find out how many things they had before them. Finding 'how many?' by grouping in tens (and then tens of tens) reminded children that our way of naming numbers uses a ten-based system, and this idea remains crucial to their understanding of the calculating techniques developed in Number, Pattern and Calculating 6, including long multiplication and division.

The second key to our symbolic numeration system is **place value**, which is a kind of shorthand describing how the place of each digit within a string of digits signifies an important value. So it is the place of '2' in the string '427' that tells us it has a value of 2 tens, or 20. It is important to realize, then, that the term 'place value' actually refers to a symbolic code for naming and reading number names, and that children have to learn either to crack the code or to reinvent it for themselves (depending on how they are taught).

Some people usefully distinguish between what is called the 'column value' and the 'quantity value' of a digit. For example, the column value of '2' in '427' is '2 tens', because it is in the 'tens' column, while its quantity value is '20', because that is its value as a quantity. In Numicon, we feel the important thing is that children understand that column value and quantity value are **equivalent**, that is, that the '2' in '427' means both '2 tens' and '20'; the two values are interchangeable. Children learn this equivalence through joining in our conversations around place value – this is another instance of the vital importance of our conversations with children when teaching mathematics.

The fact that children have spent time grouping objects in tens to 'find how many' in their early stages helps children subsequently to **partition** numbers as part of many calculating techniques. For example, seeing 236 as '200 and 30 and 6', focusing upon quantity value, can sometimes (though not always) be the most helpful way of seeing the number, and in essence this partitioning is 'undoing' the grouping that they managed earlier.

Place value in Number, Pattern and Calculating 6

Children are asked to extend their understanding of place value in Number, Pattern and Calculating 6 in two main ways: by increasing use of formal written column methods of calculating, and by extending their calculating to working with both larger and with longer numbers (that is, larger values and values expressed up to three decimal places). Both of these aspects of number work depend crucially upon generalizing from a fundamental understanding of early 'grouping in ten' activities, and of our 'place value' code for number notation.

As in Number, Pattern and Calculating 4 and Number, Pattern and Calculating 5, process terms for grouping and re-grouping in tens, hundreds and so on, such as 'carrying', 'exchanging', 're-distributing' and 're-grouping' are used explicitly. Partitioning and recombining numbers in these ways as children calculate are another instance of 'doing and undoing' actions – a key element of mathematical thinking for children to develop in many aspects of calculating (discussed further in the section on **Inverse**). Children's use of numbers up to 10 000 000, and increasing emphasis upon decimal fractions in Number, Pattern and Calculating 6 also places increasing demands upon their ability to read, order and position numbers between other numbers of any size.

As work with decimal fractions extends now to multiplying and dividing numbers with decimals, including multiplying and dividing by multiples of 10, 100 and 1000, children come increasingly to recognize and use the constant **ratio** relationships between 'places' (or columns) in our system of number notation, however large or small the numbers. That is, children increasingly recognize and use the regular '× 10' and '÷ 10' relationships between column values as we move to the left and to the right respectively in reading, writing and calculating with multi-digit numbers. The significance of the decimal point marking out 'fractions' to the right of it continues to be reinforced as children also practise rounding, and estimating the answers to calculations before calculating.

Negative numbers

Negative numbers are a very old idea, explicitly discussed and used in Hindu writings of the seventh century CE and in much earlier Chinese calculating, where the colours of 'coloured rod' images denoted either a positive or negative aspect. In both cases the context was accountancy, with negative numbers signifying an amount of debt. Calculating with these numbers was always dependent on the agreed sense they made within the practical contexts of money, loans, assets and debts.

Interestingly, for centuries many European mathematicians resisted the idea that negative numbers are as valid mathematically as 'natural' (counting) numbers, refusing to allow that negative solutions to equations could be meaningful. Children having difficulties with negative numbers today have history on their side.

In Number, Pattern and Calculating 4, following the early Chinese and Hindu examples, children were introduced to negative numbers in practical contexts in which they make sense today – in our case, temperature and underground car-parking levels. The important thing, in both contexts, is that in a real way amounts 'below zero' make intuitive and practical sense. More generally now, the idea of negative versus positive amounts often works in situations where there is some key point of reference that has meaningful amounts on either side of it, for example years before or after a key event on a timeline, or travelling towards or away from a geographical 'zero' position on a physical line.

The essential thing for children to understand as they meet negative numbers is that from now on numbers may be considered to have not just a 'size' but also a 'direction'; this is why integers (i.e. positive and negative whole numbers including **zero**) are sometimes together called 'directed numbers'.

Importantly, using a '**zero**' point on a physical line is what prepares children for the imagery of negative numbers presented as distances to the left of zero on a conventional number line. In Number, Pattern and Calculating 5, children continued to meet negative numbers in context, count forwards and backwards along a number line using both positive and negative numbers, and began to calculate differences between two directed numbers.

In Number, Pattern and Calculating 5, children were also introduced to the important idea of *ordering* directed numbers, and this raised the intriguing question of whether $^{-}12$ is 'bigger' or 'smaller' than $^{-}7$? In Number, Pattern and Calculating 6 we continue to suggest that it is always best to answer such questions in context, so that we could say $^{-}7$ °C is 'warmer' than $^{-}12$ °C, and that someone who owes £7 is 'better off' than someone who owes £12. Otherwise it is still probably best to focus simply on direction when ordering directed numbers and to ask, e.g.: 'Which is *to the right of* the other one on a number line?' In Number, Pattern and Calculating 6 children continue, in context, to calculate intervals that cross **zero** on a number line.

Factors and multiples, prime and composite numbers, squares and cubes

As children learned their multiplication tables earlier and thereby gradually became more familiar with multiplicative relationships between numbers, they were encouraged to make – and to explain – observations such as '36 seems to crop up in lots of places in the tables'. A typical explanation would be because 36 is the product of 3 × 12, 12 × 3, 4 × 9, 9 × 4, and of 6 × 6, and we were able to use observations like this to introduce children to the term **factor**, meaning a number that divides into another number exactly, without leaving a remainder. So 3, 4, 6, 9, and 12 are all called factors of 36 because they divide exactly into 36 without leaving a remainder; 2 and 18 are also factors of 36, but this is not obvious from multiplication tables up to 12 × 12. By introducing the term 'factor' we could encourage children to explain their observations with, for example, '36 seems to crop up so many times in the multiplication tables because it has a lot of factors'.

Afterwards we were also able to put the relationship the other way around and say that since 2, 3, 4, 6, 9, 12, and 18 are all factors of 36, that means conversely that 36 is a **multiple** of 2, 3, 4, 6, 9, 12, and of 18.

Not all numbers have as many different factors as 36 does; some numbers have only two different factors, and such numbers were introduced as **prime numbers**. 3, 5, 7, 11, 13, and 17 are all prime numbers; each of these is divisible only by 1 and by itself, e.g. 3 is divisible only by 1 and 3. '1' is not normally called a prime number because it only has one factor, i.e. 1 itself. 2 is the only even prime number.

Any positive whole number that is not a prime number is said to be a **composite number**. 36 is a composite number because it is a positive whole number, and it has many more than two different factors.

Any composite number can also be broken down into its **prime factors**. Working out the prime factors of any number however involves more than just listing those factors of a number that are themselves prime numbers, e.g. establishing that the only factors of 12 that are prime numbers are 2 and 3. **Prime factorization** means working out how to express any composite number as the *product* of prime factors, so for example, the prime factors of $12 = 2 \times 2 \times 3$ or $2^2 \times 3$.

By presenting composite numbers as products of their prime factors, we can make some calculations easier, or more systematic. In particular, 'products of prime factors' make finding the Highest Common Factor (HCF) of two numbers, or their Lowest Common Multiple (LCM) easier, and also the reducing of common fractions to their lowest terms.

Factors and multiples, prime and composite numbers, squares and cubes in Number, Pattern and Calculating 6

In Number, Pattern and Calculating 6, children are invited to make use of their increasing familiarity with tables facts (i.e. factors and multiples), as they learn to 'simplify' fractions, to convert improper fractions to mixed numbers (and vice versa), and to calculate equivalent fractions in order to compare them, and order them, and to add and subtract them from each other.

Children continue to recognize that when a number is multiplied by itself, the product is said to be a **square number** (probably because the area of a square is calculated by multiplying the length of its side by itself). As an example, 36 is called a square number since it is the product of 6×6.

Children also continue to recognize that when a number is multiplied by itself twice, the product is said to be a **cubic number** (or a '**cube**'), probably because the volume of a cube is calculated by multiplying the length of its side by itself, twice. For example, 8 is a cubic number since it is the product of $2 \times 2 \times 2$.

In Number, Pattern and Calculating 5 children also were also introduced to tests of what is called 'divisibility'; these are ways of testing whether a number has a particular factor, or not. For example, a number has a factor of 4 if the last two digits are also divisible by 4. Children continue to explore such patterns in Number, Pattern and Calculating 6 as part of their continuing work on recognizing factors, multiples, and prime numbers more fluently.

Fractions, decimals, ratios, proportion and percentages

Fractions occur as part of a complex set of relationships, and confusingly for many children there are also several different symbolic ways of representing what are essentially the same numbers, e.g. $\frac{3}{5} = \frac{9}{15} = 0{\cdot}6 = 60\% = 3:5 = 3 \div 5$. One of the key challenges for teachers at this stage is to guide children to understanding that common fractions, decimal fractions, percentages, ratios, proportions and dividing calculations are essentially different forms of notation for expressing the same 'rational' numbers, and that **ratio** is at the heart of what is called overall, **multiplicative thinking**.

Typically for children, fractions *of* things arise in measuring situations, which importantly include 'sharing'. The measuring of continuous quantities, such as time, length, or chocolate and so on is always approximate and for this reason we commonly find ourselves needing parts of whole units to describe amounts accurately. The moral imperative for fair shares usually draws children easily to the view that fractions are, and indeed should be, about *equal* parts (or **proportions**) of a whole (although in Number, Pattern and Calculating 6, as children study proportion, they will also meet instances of unequal sharing).

The two main ways in which children communicated about fractions in early Numicon activities were as 'operators' and as 'descriptors' – using fraction words as verbs and as adjectives. An initial invitation to '*halve* twenty-six' would be an invitation actively to find 'half' of 26 – the fraction word is used initially as part of an instruction to *do* something. Then, to describe the outcome of some measuring tasks, or of

some dividing calculations, children would use fraction words as adjectives, for example in the description 'twenty-six-and-a-half *somethings*', or as the description of a relative distance, for example as 'halfway' between 26 and 27 on a measuring scale.

It is important to note too that in work from the *Number, Pattern and Calculating 2 Teaching Resource Handbook* onwards, children were also meeting fractions as *objects* (that is, as 'numbers' in themselves), signalled by the use of fraction words as *nouns* and by their representation along a pure number line. From introducing halving situations in which there was an action of halving (say) a pizza, and in which children used the verb, 'to halve', we moved to describing actual amounts of things using fraction words as adjectives; then we subsequently asked them to do a very strange thing, which was to start talking about 'a half' as a rather isolated abstract mathematical object, using the word 'half' on its own, as a noun. Note that in a mathematical context, in which fraction words are used as nouns, any trace of pizza (or of anything else from a material world) has disappeared altogether; the key illustration that supports children thinking about '$\frac{1}{2}$' as an abstract mathematical object is usually the distance along a number line between 0 and the 'half-way' point between 0 and 1.

It cannot be emphasized too strongly that talk of actions (using verbs), turning to talk of 'fractions *of* something' (using adjectives) turning to talk of just isolated 'fractions' as mathematical objects (using nouns) involves very significant changes in our communicating for children to join in with. The models and imagery we offer children to help them get used to these strange new aspects of our communicating are crucial, and once again children need plenty of time, opportunity and imagery to get fully used to our new ways of talking and communicating about them.

When using fraction words as nouns and beginning to ask children questions such as '$\frac{2}{5} + \frac{4}{5}$ = ?', it can be helpful to suggest that children read the number sentence as, '$\frac{2}{5}$ *of anything* + $\frac{4}{5}$ *of anything* = what?' This may help children to understand that fraction words and symbols used on their own, as nouns, are **generalizations**, and that '$\frac{2}{5}$' used on its own means '$\frac{2}{5}$ *of anything*'.

In Number, Pattern and Calculating 1 and 2, common (or 'vulgar') fractions were introduced with their conventional notation (e.g. '$\frac{2}{3}$'), and as new number objects they began to be related to existing whole numbers through also representing them as distances along a number line. In Number, Pattern and Calculating 3, the terms 'numerator' and 'denominator' were formally introduced, counting on and back in fractions along a number line was further developed, and some fractions (< 1) with the same denominator were **added** and **subtracted**.

In Number, Pattern and Calculating 4, key developments involved the introduction of decimal fractions, mixed numbers and improper fractions and, importantly, recognizing the **equivalence** of a range of common fractions (< 1) e.g. $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$ …, which is fundamental for later methods of adding and subtracting fractions with different denominators.

Importantly, strong explicit connections were made in Number, Pattern and Calculating 5 between common fraction notation and the process of dividing, that is, recognizing that $\frac{5}{8}$ is **equivalent** to 5 ÷ 8. It can, however, take some time for children to accept an equivalence between an action (dividing) and an object (a fraction), and it will be useful still to allow plenty of discussion about this in Number, Pattern and Calculating 6 as well.

We discuss the important relations between the **fractions, ratios**, and **proportions** considered in Number, Pattern and Calculating 6 in the section **multiplicative thinking**.

Fractions, decimals, ratios, proportion and percentages in Number, Pattern and Calculating 6

In Number, Pattern and Calculating 6, children continue to develop their abilities to translate fluently between the various forms of fraction notation: between equivalent common proper fractions, between mixed numbers and improper fractions, and between common fractions, decimals, percentages, and division calculations. This developing flexibility of expression allows children now to add and subtract fractions with different denominators.

Significant further developments in Number, Pattern and Calculating 6 are the introduction to multiplying proper fractions, and to dividing proper fractions by a whole number. The illustrative use of arrays and rectangles is crucial to children's understanding of how to multiply proper fractions together, and the dividing of proper fractions by whole numbers can be well supported by connections with the dividing of decimal fractions by whole numbers.

The use of Numicon Shapes, number rods and objects arranged in arrays, and visual imagery (such as diagrams and number lines) continues to be essential to communicating about fractions, as is the use of everyday and realistic contexts that children can relate to. Measuring scales are particularly useful. When using the terms 'numerator' and 'denominator' with older children it can also be helpful to explain their sense. A de**nom**inator gives a common fraction its name – it tells you what kind of a fraction it is. A **num**erator tells you how many of this kind of fraction you have. There is always a history to how we do and say things in mathematics.

Complex expressions involving arithmetic operations – BODMAS

As the mathematical problems children face become more complex, so calculating becomes more complex in that several arithmetic operations can be combined together in one sequence. For example,

$$T = 5\ (3^2 + 8 \times 7) \div 6\ (4 + \sqrt{9} \times 8)$$

Since the above expression for 'T' could be simplified, or T's value could be calculated, in a number of different ways that would all give different answers, there is an agreed international convention that determines the order in which the various operations involved are to be tackled, so as to avoid ambiguity. In the UK, this convention is usually remembered with the help of the acronym BODMAS, which identifies the sequence of operations as:

Brackets (do everything inside brackets first)

Order (that is, powers and roots)

Division and **M**ultiplication

Addition and **S**ubtraction.

Also, when two or more operations of the same order appear one after another, the operations should be carried out from left to right. So,

$$24 \div 3 \times 4 = 32 \text{ (not 2)}$$

So, we would begin to deal with the expression for T by looking inside the brackets first (**B**), and within the brackets dealing first with powers and roots (**O**). Purely for the purposes of this example, if we take $\sqrt{9}$ as $^+3$ (and ignore the possibility of $^-3$ as another square root), this gives us:

$$T = 5\ (9 + 8 \times 7) \div 6\ (4 + 3 \times 8)$$

We can then finish off the work inside the brackets by doing the multiplying (**M**) before we do the adding (**A**):

$$T = 5\ (65) \div 6\ (28)$$

Which would then become:

$$T = 325 \div 168$$

As a division, this can then be resolved into whatever form suits the context of the original problem – either as an improper fraction, a mixed number, a decimal fraction, or leaving a remainder.

BODMAS priorities also resolve some quite subtle associated ambiguities that become more important as children progress, for example: -5^2 is conventionally interpreted to mean $-(5^2)$, meaning 'do the power first', giving a value of $^-25$, rather than $(^-5)^2$ which would be 'do the negative first', giving an answer of $^+25$.

Two further aspects of the BODMAS convention are important for children's future progress: not all electronic calculators (or indeed programming languages) follow this convention (some calculators just do the operations in the order you key them in), and this convention underlies the interpretation of algebraic expressions and formulae that children will meet later in secondary school, such as the formula for solving quadratic equations:

$$x = \frac{^-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The order in which these calculations are done will usually make a big difference to the result.

Arithmetic operations, or 'the four rules': adding and subtracting

In Number, Pattern and Calculating 6, children continue to develop their adding and subtracting of whole numbers, with an emphasis upon written methods necessary for calculations too difficult to be accomplished purely mentally. Note that all written calculating involves an element of working mentally (remembering number facts and so on), and children should therefore also practise mental arithmetic continually.

Even though much focus is upon developing written column methods of calculating in Number, Pattern and Calculating 6, children should also always be encouraged to think about any calculation first before simply diving in thoughtlessly with the first method that occurs to them. **Non-computational thinking** may often reduce an apparently difficult calculation to a much easier one that can be carried out purely mentally. Much practice at consciously thinking about how to approach any particular calculation before diving in is essential at this stage, as children develop the flexibility in their calculating that is necessary for fluency.

This is an important point and worth emphasizing: the use of base-ten materials is purely to *illustrate* number **equivalence** relationships in ways that reflect our **place value** system of naming numbers. Actions with the materials are used therefore simply to help children think about how the 'grouping' or 'exchanging' actions involved in most written methods *make sense*, not as a method of 'calculating with blocks' in itself. Actions with materials are not carried out to *produce* an answer, but to *explain* the number actions involved in calculating.

In Number, Pattern and Calculating 6, children build on their knowledge of adding and subtracting both common fractions and decimal fractions, and these operations continue to rely upon children's understanding of both **equivalence** and of **place**, as they now work on adding and subtracting common fractions with *any* denominators.

With all kinds of numbers, with positive and negative whole numbers, and with common and decimal fractions, children continue to work on the following aspects of adding and subtracting in the Number, Pattern and Calculating 6 activities:

- **structures** – the different kinds of situations in which adding and subtracting occur; and
- **methods** – how to calculate.

Structures for adding and subtracting

Within Numicon, we address two adding structures – **aggregation** and **augmentation** – and children should be given regular experiences with both forms in a variety of contexts.

Aggregation is putting together. Two or more amounts or numbers are put together to make a 'total' or 'sum'. For example: 'I had £20. John gave me £10 and Nana gave me another £35'. How much did I have in total?'

Augmentation is about increase. One amount is increased or made bigger. For example: 'Special offer! One third extra free!'

We expect children to recognize four subtracting structures: **take away**, **decrease**, **comparison** and **inverse of adding**. Subtracting is more complex than adding as it is more varied in the different kinds of situations in which it occurs. Again, children should be given experience with all four structures regularly.

Take away refers to those situations where something is lost, or one thing is taken away from another. For example: 'Gemma had £19 at the beginning of the day. She spent £6·47. How much does she have now?'

Decrease is about reduction. For example, 'Special offer! 25% off!'

Comparison occurs when two amounts are being compared and we want to find the additive difference. For example, 'Samir has saved £34·40 and Nihal has £42·65. What is the difference between the amounts of money that Samir and Nihal have?'

As comparisons involving negative numbers are addressed in Number, Pattern and Calculating 6, comparisons and differences between ranges of positive and negative numbers are most effectively illustrated using a continuous number line.

The **inverse of adding** structure is about wanting to know how much more of something we want or need in order to reach a particular target. For example, 'The blue trainers cost

£59·50. I have £38·25. How much more do I need to buy the shoes?' Children can often feel very confused about adding on in order to accomplish subtracting, and it is important for teachers to be clear about what is going on here. The reason this adding manoeuvre is included as a subtracting structure is because the adding on in these cases is done in order to find out a *difference*; in most adding we know how much to add, and we do it.

Methods for adding and subtracting in Number, Pattern and Calculating 6

There is continuing emphasis in Number, Pattern and Calculating 6 on developing written methods of adding and subtracting with larger numbers, but children should always be encouraged to think before they act. In particular, children should always be encouraged to think first about whether any given calculation could be transformed into an easier, equivalent calculation, and secondly to estimate what the answer is likely to be – approximately – before calculating.

Non-computational thinking should thus become a habit as children are increasingly asked to think about their calculating, rather than just responding mechanically to an addition or subtraction sign. Is there the possibility of altering a calculation to an equivalent 'easier' one, for instance altering 840 – 380 to (840 – 400) + 20? (Adjustments of this particular kind are sometimes called 'rounding and compensating'.) Children should regularly be encouraged to notice how useful the basic number facts to ten are in calculating with larger numbers, for example in being able to generalize from 6 + 4 = 10 to 60 + 40 = 100, and to 600 + 400 = 1 000.

Non-computational thinking and estimating answers to calculations in advance should of course become a habit for children when using *any* kind of numbers, but especially when adding and subtracting fractions and decimals. This will help to develop their understanding of these numbers enormously.

When adding and subtracting fractions children will find that just as with currencies of different denominations, it is not possible to add (or subtract) fractions of different denominations. In the same way that we cannot add or subtract $6 and £3 together directly, so we cannot add or subtract $\frac{4}{5}$ and $\frac{3}{8}$ together directly; in both cases we have to transform the amounts into *equivalent* amounts in a *common* denomination. We either have to convert $6 to GBP (or £3 to $), or convert both to a common third currency (e.g. €) before we can add or subtract them; with fractions we would normally convert each of the two above to equivalent numbers of fortieths (their lowest common denominator).

When adding and subtracting numbers involving decimal fractions, children's existing understanding of the place

value notation they use for naming whole numbers will be the basis from which they *generalize* to bring meaning to the columns to the right of the decimal point. In particular, children will need to be clear that the 'value' of a column 'place' divides by ten each time we move one 'place' to the right. Thus the columns to the right of the decimal point have values of one tenth, one hundredth, one thousandth, and so on as we move to the right.

In Number, Pattern and Calculating 5, children began usefully to generalize the technique of '**bridging**' through multiples of ten to bridging through larger multiples (e.g. 100), and also to bridging through different kinds of convenient 'whole' points in different contexts. For example, when adding $\frac{4}{5}$ and $\frac{3}{5}$ together we can mentally partition the $\frac{3}{5}$ into $\frac{1}{5}$ and $\frac{2}{5}$, add the $\frac{1}{5}$ on to the $\frac{4}{5}$ first, and thus 'bridge' through 1 to obtain the answer '$1\frac{2}{5}$'. Note that we could also 'bridge through 1' if we were doing the same calculation with decimal fractions: '0·8 + 0·6' can be managed as 0·8 + (0·2 + 0·4), which is the same as (0·8 + 0·2) + 0·4, which then gives the total '1·4'. Bridging was thus introduced as an invaluable mental technique for adding and subtracting in a wide range of contexts, and particularly when using the units of various measures as bridging points; when adding 350 m to 1 km 800 m for example, it is helpful to 'bridge' through 2 km. This technique should be regularly encouraged in Number, Pattern and Calculating 6 as it can add significantly to children's calculating fluency, particularly when problem solving.

Arithmetic operations, or 'the four rules': multiplying and dividing

In Number, Pattern and Calculating 6, work also continues on straightforward multiplying and dividing situations and calculations with larger whole numbers, using both purely mental and written methods. As with adding and subtracting, children should be encouraged always to think before they act; **non-computational thinking** will often reveal ways of making an apparently difficult multiplying or dividing calculation much simpler, and children should also get into the habit of always estimating what an answer is likely to be, approximately.

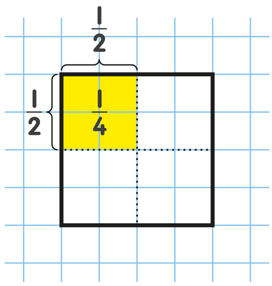
The calculation 36 × 25 = ☐ for example, can be re-interpreted as (9 × 4) × 25 = ☐ using a particular pair of **factors** of 36, and then as 9 × (4 × 25) = ☐ (using the **associative property**), which is then easy to calculate purely mentally. In this case the 'non-computational' recognition and use of factors removes the need for any laborious form of 'long' written multiplication.

Also, as with children's work on adding and subtracting whole numbers, base-ten apparatus is often used in Number, Pattern and Calculating 6 to illustrate the number relationships involved in 'partitioning', 'exchanging', 'grouping' and so on that feature within some written methods of multiplying and dividing with larger numbers. It is again important to establish that base-ten materials – as with all other materials and imagery used in our approach – are used to *illustrate relationships*. Actions with physical materials are not used as methods of producing answers, but to *explain* the sense of number actions that are used in calculating with figures on a page.

By the end of Number, Pattern and Calculating 6, children should be accomplished in using both long multiplication and long division column methods effectively; when either the numbers involved are too awkward, or when non-computational thinking cannot translate a multiplying or dividing calculation into an easier form, children should be able to employ both long multiplication and long division methods smoothly. This takes both understanding and practice.

In Number, Pattern and Calculating 5, children also began to multiply whole numbers with both common fractions and with decimal fractions, and this placed considerably more demand upon their **multiplicative thinking**. 'Multiplicative thinking' is a term increasingly used now to refer to a whole set of different ways of thinking about *comparison relationships*, and is usually contrasted to 'additive thinking' (see below).

In Number, Pattern and Calculating 6, children are asked to multiply proper fractions together, and also to divide proper fractions by whole numbers. Multiplying proper fractions together often produces surprises for children, who are used to multiplying whole numbers together and obtaining larger products; many children cannot understand how, for example, $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, in which the product is smaller than either of the two numbers being multiplied. The key illustration supporting this work is that of a rectangular array, most helpfully drawn on squared paper.



Dividing proper fractions by whole numbers produces smaller outcomes, and this seems generally more acceptable intuitively.

Multiplying and dividing

The operations of multiplying and dividing have several aspects, not all of which make coherent sense to children for a considerable time. In Number, Pattern and Calculating 6, we continue to distinguish between the **repeated adding**, **ratio** (or **scaling**) and **array** structures of multiplying. Repeated adding and scaling up are often fairly intuitively understandable and build on children's earlier experiences of counting on in 2s, 5s and 10s, and doubling in Number, Pattern and Calculating 2 and 3. The array structure becomes increasingly important in Number, Pattern and Calculating 6 as children learn to generalize their multiplying to a wider range of numbers, particularly to fractions.

As with our discussion of adding and subtracting, in relation to multiplying and dividing we address:

- **structures** – the different kinds of situations in which multiplying and dividing occur; and
- **methods** – how to calculate.

Structures for multiplying

Repeated adding is the familiar 'so many lots of something' idea, in which repeated equal amounts are added. For example, '5 tables each need 6 place settings. How many place settings are needed altogether?'

Ratio is the 'multiplying up' idea we use when we want to scale something up, for example, making a recipe for 6 people instead of 2.

Both of these structures have very important and strong **inverse** connections with **dividing**. Scaling up, for instance, is associated with its inverse in dividing, of scaling down – for example, halving.

Importantly, children should be encouraged to notice that when two numbers are being multiplied together to give a product, in many situations each number plays a different role – one number refers to an amount being multiplied (technically, this is the 'multiplicand') and the other determines how many times that number is to be multiplied (the 'multiplier'). One number is being multiplied; the other does the multiplying. In practice, we teach children pretty quickly that multiplying has a **commutative property** so, for example, $4 \times 6 = 6 \times 4$ and that in a sense it doesn't matter which number is doing the multiplying, as the product will be the same.

However, this practice of saying 'it doesn't matter which one is the multiplier' can turn out to be unhelpful to children when they later try to make sense of the **inverse** connections between multiplying and dividing. Seeing dividing as the inverse of multiplying (and using our multiplication tables to solve dividing problems) is a bit like turning a dividing problem around and saying, 'We already know the product of two numbers, but we only know one of the numbers that were multiplied.'

And in practical situations, it can make a difference whether the number we know is the multiplicand or the multiplier. If we know the multiplicand (the size of the groups), we then want to know how many times that number goes into the product; if we know the multiplier, or how many times something goes into the product, we want to know how big that 'something' (the multiplicand, or size of the group) was. The first case applies to dividing situations like working out how many 15-seater minibuses we need to ferry 60 children around (this is called 'quotition'). The second applies to sharing situations such as 'How much will we each get of that cake?' (this is called 'partition').

The third multiplying structure, that of an **array**, will help children to see the **commutative** property of multiplying,

and in Number, Pattern and Calculating 6 it also helps them to connect multiplying with the measurement of area, to understand how multiplying by fractions makes answers smaller, and to understand how (in long multiplication) the multiplication of large numbers can be broken down into smaller calculations (the **distributive property**). All of this involves interpreting multiplying as an 'array', for example illustrating 3×4 as:

• • • •
• • • •
• • • •

In Number, Pattern and Calculating 4, we placed increased emphasis on arrays as children worked to generalize the distributive property – that is, that multiplication is 'distributive over addition'. It is this property that underlies the 'grid method' of multiplication that was also introduced at that stage (see the section on the **distributive property**). Arrays are also helpful to children when facing **correspondence problems** such as, 'I have 5 T-shirts and 3 hats; how many different outfits can I put together?' (see the section on **multiplicative thinking**).

The **ratio** structure of multiplying was developed in the *Number, Pattern and Calculating 3* and *4 Teaching Resource Handbooks* in terms of everyday situations that required some scaling up, for example, of recipes. In order to establish essential links with dividing from the beginning, the images and patterns developed in Numicon activities for multiplying are usually quickly exploited to illustrate dividing; recipes are also scaled down to offer a context for the **ratio** structure of dividing.

All the multiplying structures continue to be relevant in Number, Pattern and Calculating 6, especially as children are increasingly asked to solve problems in a variety of contexts, but (as noted above) the array structure is probably the illustration that best supports the key developments in Number, Pattern and Calculating 6.

Note on a reading convention: When reading and recording multiplying sentences such as '4 × 7 = 28', there are many choices of interpretation and often a surprising amount of controversy about whether '4 × 7' *really* means 'four 7s' or 'seven 4s'. Of course, the array structure quickly demonstrates that their product is the same, but some teachers feel that only one reading of the sentence can be 'mathematically correct'.

The truth is that we do have choices, and that there are equally good reasons for choosing either way. In Numicon activities, we have chosen to introduce reading '4 × 7' as 'four times seven', meaning four lots of seven, for a number of reasons: in order to exploit the everyday use of the word 'times' (signalling repeated actions); to tie in with the traditional way of reading and saying multiplication tables in the UK; to be consistent with conventions for units of measure – for example with the meaning of 3 kg as three 'lots of' a kilogram; and to be consistent with algebraic expressions such as $4x + 3y$ (commonly interpreted as '4 lots of x' and '3 lots of y').

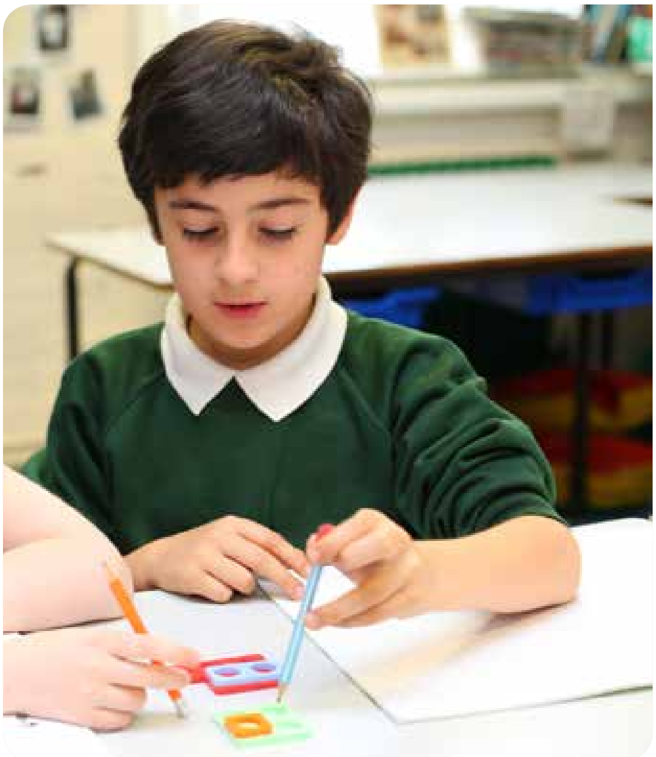
Structures for dividing

There are three essential structures of dividing: the **grouping**, **sharing**, and **ratio** (or **scaling down**) structures.

The **grouping** structure – technically called quotition – occurs in situations where we know an amount, the dividend, and we want to know how many times a different amount, the divisor, will go into it. This type of situation will lead to remainders when the divisor is not a **factor** of the dividend. For example, 8 goes into 43 five times, leaving a remainder of 3. We often call this '8 divided into 43' (or '8s into 43'). It is the grouping structure that underlies the form of long division described as 'chunking'.

The **sharing** structure – technically called partition – occurs in situations where, again, we know the amount to be shared, the dividend, but this time we know how many equal parts the dividend is to be shared into but we don't know how big each share will be. This type of situation will lead to **fractions** when the number of shares, the divisor, is not a **factor** of the dividend and the object(s) being shared can be broken into parts. For example, 3 chocolate bars shared between 2 people will give each person $1\frac{1}{2}$ bars. We might call this '3 divided into 2 parts' (hence sharing is called partition). Note: this type of situation may also help children to understand the **equivalence** of $\frac{3}{2}$ and '3 ÷ 2'. The sharing structure often offers a clearer explanation of traditional long division method, particularly if base-ten blocks are used as illustrations (see Calculating 10).

It is important that children learn to distinguish between these two types of situation if their answers to dividing



problems are to make sense. If we have a situation where we have some money, for example £32 (the dividend), and we want to know how many tickets each costing £1·50 we can buy with the money, the answer is 21 remainder 50p, not (as a calculator would show) 21·3333 … . This is a grouping (or quotition) problem; we want to know how many times £1·50 goes into £32 – there is no sharing involved and fractions make no sense as an answer. This is £1·50 into £32.

On the other hand, if 3 children are going to share 10 fish fingers (the dividend) between them (fairly!) their equal shares will be 3·3333 … ($3\frac{1}{3}$) fish fingers each. In this situation (partition), we already know how many parts the fish fingers are to be divided into (3), but we don't know how big the resulting equal parts will be. This is 10 shared into 3 equal parts.

Unfortunately for children (once again), the language we use when speaking of dividing calculations is often confusing. As you may already have noticed, we tend to use the word 'into' for both sorts of situation – 'dividing 3s into 10' (quotition), as well as 'sharing 10 into 3' (partition). We also often speak of 'dividing 10 by 3'. To complicate things further, when we introduce mathematical symbols for dividing to children, we often tend to imply different structures for the calculation rather carelessly. For instance, 10 ÷ 3 tends to be read as 'ten divided by three' and is often explained as a 'sharing' problem, whereas $3\overline{)10}$ tends to be read as 'threes into ten', probably because (reading conventionally from left to right) the numbers in this second case appear in the reverse order and we want children to use their tables as they solve it. In both cases we want the children to do the same dividing calculation, but these symbols are often explained as two quite different structures of dividing (the first partition, and the second quotition) and children can very reasonably struggle to understand exactly what it is we want them to do. Do we want them to share 10 into 3 (partition), or to find how many 3s in 10 (quotition)?

Eventually children will come to understand that dividing can be seen either way, but that the way we see a dividing problem will affect the kind of answer we give. Mixing up both structures of dividing without making them distinct to children often leads to confusion, and to answers that don't make sense. In particular, children often struggle to understand what to do with remainders (see below).

The **ratio** structure of dividing occurs when something is being scaled down, for example when a scale model – usually of something large – is made. The classic examples are of course maps, in which large actual geographical distances are all divided by the same number to produce an image that fits onto a piece of paper.

Numicon activities first introduced dividing as grouping (quotition) and left sharing (partition) and fractions resulting from a dividing calculation for subsequent activities. This was done in order to put some mental space between initial experiences of the two structures. Simple **fractions** were of course discussed with children earlier, but usually in the context of incomplete units of measure, and not as ways of dividing up whole-number remainders of division calculations.

In Numicon activities, we have always introduced dividing firstly and distinctively as the grouping structure, which emphasizes its **inverse** relation to multiplying. Having shown how 7 × 3 = 21 in multiplying, we then connected this with dividing as grouping (quotition) by asking, 'If 7 times 3 is 21, how many 3s are there in 21?' In Number, Pattern and Calculating 6, we continue to emphasize the **inverse** relationship between dividing and multiplying at every opportunity.

Finally, the **ratio** structure of dividing was first introduced in Number, Pattern and Calculating 1 with halving, and subsequently children have also been invited to find thirds, quarters and sixths of various quantities. This can also be seen as multiplying by a half, a third, a quarter, and so on (again, see the section on **inverse relationships**), but children are not asked to make this connection with multiplicative inverses explicit yet. Contexts featuring the **ratio** structure of dividing continue to feature in Number, Pattern and Calculating 6 activities.

All dividing structures continue to be relevant to Number, Pattern and Calculating 6 work, especially as children are invited more and more to decide for themselves how to solve problems in context, and thus to decide which arithmetical operations to apply in a situation, and when.

A special note about remainders

When a dividing calculation involving whole numbers doesn't work out exactly, we divide the dividend by the divisor so far as we can, and then find we have a small whole number 'left over'. Children are often confused about what to do with the leftover number; sometimes we leave it as a 'remainder', and sometimes we carry on dividing the remainder into fractions (or decimals). When should we do which? The answer always depends upon the context in which the dividing has arisen, and often the difficulty for children is that there are two quite different reasons for leaving a remainder and a third reason for going into fractions.

The first reason for leaving a remainder is that we are dealing with a quotition situation and fractions would not make sense. If 23 people need taxis home and a taxi will take five people, we divide 5s into 23. The solution to this problem is not that we need $4\frac{3}{5}$ taxis, but that hiring 4 taxis will leave a remainder of 3 people unable to get home (so we'd better order 5 taxis).

The second reason for leaving a remainder is totally different: in a partition situation, we might find ourselves sharing out objects that cannot be broken into smaller parts. If 3 children have 50p to share between them, they can have 16p each and there will be 2p left over that cannot physically be broken down into three equal parts – it therefore has to be left as a remainder.

The reason for continuing to divide a leftover whole number into fractions occurs in a partition situation in which the thing being shared can actually be broken down into smaller parts. (The word 'fraction' comes from the same root as the word 'fracture'.) If 6 people are sharing 4 pizzas (4 ÷ 6 as a dividing problem), the solution is not 0 remainder 4 pizzas (everyone gets nothing), but that each person gets $\frac{2}{3}$ (or $\frac{4}{6}$) of a pizza. Note again the usefulness of a situation like this for illustrating the **equivalences** between '4 ÷ 6' and $\frac{4}{6}$ and $\frac{2}{3}$.

Children continue to need much experience with and illustrated discussion of all three kinds of situation in Number, Pattern and Calculating 6.

Methods of multiplying and dividing

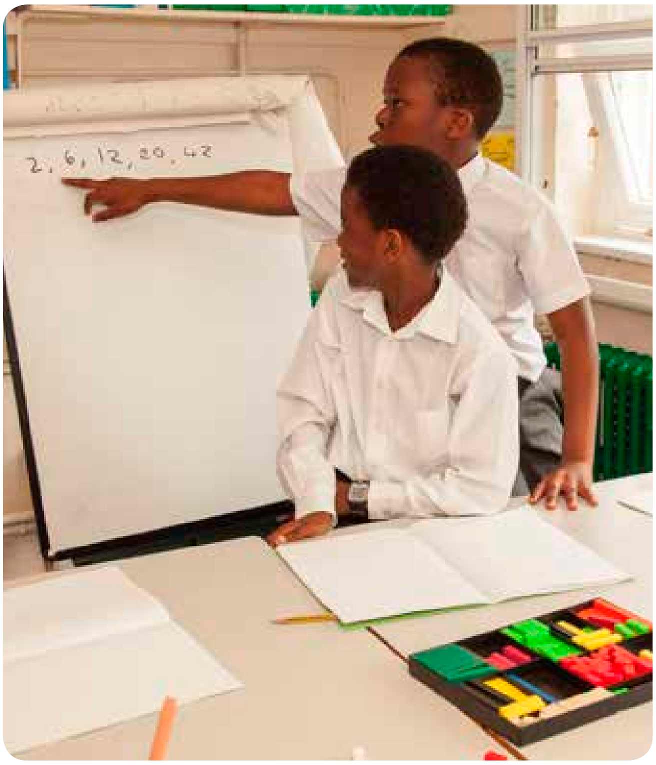
As with adding and subtracting, we work on developing children's fluency in multiplying and dividing primarily through ensuring that all work is grounded in a depth of understanding of the natures of these operations, of the types of contexts they are relevant to, and of how (in our base-ten system) both numbers and operations 'fit together'.

As with adding and subtracting, when multiplying and dividing, children should get into the habit of always asking first whether **non-computational thinking** might allow them to change any calculation into an easier (or more convenient) form, and also to estimate what any product or quotient is likely to be, before calculating. For example, $19 \times 45 = \square$ can be thought of as $(20 \times 45) - 45 = \square$, and then as $900 - 45 = \square$, which we would by now expect children to do mentally. Such thinking and estimating is particularly important when dealing with fractions and decimals, to support children in building their understanding of **multiplicative thinking**.

In a scaling down situation, such as modifying a recipe for fewer people, children might need to find a third of 250 g of flour, i.e. '$\frac{1}{3} \times 250$ g'. It will probably help them to be able to translate this multiplying calculation into the division $250 \div 3$, to do the dividing, and then to reason that neither 83·3333 … nor '83 and one left over' are helpful answers in this context, so that $250\text{ g} \div 3 \approx 83\text{ g}$ (or even ≈ 80 g) will be the most sensible answer. This is an example of what is called **multiplicative thinking** in context, even though the calculating involved turned out to be dividing.

Children need to learn their multiplication tables, and in the Numicon activities these essential facts have been introduced in an organized sequence, exploiting available patterns along the way to ensure the tables are both connected with each other and memorable. Use of multiplication tables allows children to cope with multiplying small or easy numbers mentally and (as with adding and subtracting) mental methods should always be considered as a first resort. Recalling tables facts quickly is of course essential to fluency with all written methods of multiplying and dividing.

Written column methods of short multiplying and dividing were first developed in Number, Pattern and Calculating 4, and were then extended in Number, Pattern and Calculating 5 to include decimal fractions. The 'grid method' of multiplying – crucially built upon experience with visual arrays – was continued as well, and in Number, Pattern and Calculating 5 this was then complemented by the more traditional 'long multiplication' method, both methods in fact reinforcing understanding of the **distributive property**, albeit in different ways.



There is a continuing emphasis upon multiplying and dividing by increasingly large multiples of 10 in Number, Pattern and Calculating 6, partly as the easiest examples of scaling up and down to calculate (in our base-ten system), and partly also to support children's extension of whole number multiplying and dividing into multiplying and dividing with decimal fractions. Children are now introduced to column methods of long division, and multiplying and dividing more generally with numbers that can involve calculating up to two decimal places in answers.

In Number, Pattern and Calculating 5, children also began multiplying common fractions, decimals, and mixed numbers by whole numbers. These calculations are often important within measuring contexts involving derived measures, such as finding the paved area of a pathway $7\text{ m} \times \frac{1}{2}\text{ m}$. This work is also importantly connected with multiplying and dividing to find *fractions of* particular amounts, that is, recognizing the **equivalences** between '$\frac{5}{8}$ of 23', '$\frac{5}{8} \times 23$' and '$(5 \times 23) \div 8$'.

In Number, Pattern and Calculating 6 children are introduced to multiplying proper fractions together, and although the actual calculating process is (by now) easy for children, making sense of the results generally does depend upon an array illustration (see illustration of $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, above). Children are also introduced to dividing proper fractions by whole numbers in Number, Pattern and Calculating 6, e.g. $\frac{1}{6} \div 2$, and there are some interesting possibilities to explore with children about how to do these calculations. One useful avenue to explore here involves investigating the **equivalence** between dividing by 2, and multiplying by $\frac{1}{2}$, and so on.

Multiplicative thinking

There is no doubt that what has come to be called by many people 'multiplicative thinking', or sometimes 'proportional reasoning', is a distinctive way of thinking about relationships that underlies much later work for children, both in mathematics and in science. It would be fair to say that without developing their multiplicative thinking, children will be unable to cope with large swathes of their secondary schooling, or indeed with many crucial aspects of their everyday and working lives. Our physical universe and our social worlds are each filled with multiplicative relationships that we need to be able to manage.

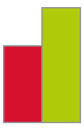
To name a few, sharing, cooking, making drinks, preparing medicines, regulating doses, maps, models, making predictions, assessing risks, measuring speeds, anticipating income, planning spending, engineering, doing science, converting currencies, designing anything, and comparing performances all use multiplicative thinking.

The root idea behind multiplicative thinking is that of a **ratio** comparison (or correspondence), and this is typically contrasted to additive comparisons. There is some evidence that children typically latch on to additive comparisons (He's got more than me! 6 = 3 + 3) significantly before they learn to make multiplicative comparisons (He's got twice as much as me! 6 = 2 × 3), and that ratios and **proportions** are almost universally much harder for us to think about than additive relationships are.

It is important to remember that although this type of thinking is called 'multiplicative', it often involves dividing. This is because multiplying and dividing are essentially but 'two sides of the same coin'. As **inverse** operations, each one may 'undo' the other, and almost every dividing problem can be converted into an **equivalent** multiplying problem, for example $17 \div 6 = 17 \times \frac{1}{6}$. (Dividing by zero cannot be converted into a multiplying problem, which is why we say it has no mathematically valid answer.)

Ratio

A ratio is a *multiplicative* comparison (or correspondence) between two values of the same kind, usually expressed in terms of one of the values being 'x times as much as' the other. Most importantly, the 'number of times' does not have to be a whole number, so comparing a 2 cm length with a 3 cm length multiplicatively for example, we could say that 3 cm is '$1\frac{1}{2}$ times as long as' 2 cm, or that 2 cm is '$\frac{2}{3}$ as long as' 3 cm. Conventionally, we could write in ratio notation that they are in the ratio 2 : 3 or 3 : 2 to each other, depending on which of them we choose to put first.



Closely connected with the idea of a ratio is that of a **rate**; a rate relates two measurements of *different* kinds. Speed is a relationship of distance to time, e.g. 30 miles per (or 'for every') hour. Many (but not all) important rates involve time as one of the measurements because time is such a fundamental aspect of our universe, but currency exchange **rates** are often quite important to us as well in everyday life.

Proportion

The term 'proportion' is used in more than one way unfortunately, but it is perhaps most often used to judge whether two or more sets of values have the same **ratio** to each other within each set. So, in mixing up two separate glasses of orange squash, if they both have the same strength, i.e. the same **ratio** of cordial to water as each other, then the two mixtures are said to be **in proportion** (or **proportional**) to each other. This leads to a form in which the idea of **proportion** is often presented, as a relationship between four values:

$$\frac{a}{b} = \frac{c}{d}$$

In words, this might be read as, 'the **ratio** of a to b is the same as the **ratio** of c to d'. In context, this might come out as, 'the **ratio** of cordial to water in this glass is the same as the **ratio** of cordial to water in that glass; the two mixtures are in the same **proportion**'. Problems involving proportions are commonly situations in which three of the four values are known, and the fourth is to be calculated.

(Notice again how fraction notation is sometimes used to specify ratios. This is because there is an **equivalence** between saying, for example, 'these two lengths are in the ratio 2 : 5 to each other', and 'the first length is $\frac{2}{5}$ of the length of the other'. Fractions are often called 'rational numbers'.)

Another use of the word 'proportion' is in contexts where values change in relation to each other; in such situations, a relationship may be either in direct or inverse proportion. In science, mass and weight are said to be directly proportional to each other; this means that if you double the mass of something, you will also double its weight. This directly proportional relationship is what allows us to compare masses in practice by comparing weights.

Scaling up or down is another important context within which proportions have to be directly maintained, for example in recipes, maps, and in geometrical transformations. Doubling a distance on a normal map represents double the distance on the actual ground.

In another context, speed and the time needed for a journey are *inversely* proportional to each other; if you double your speed for a journey, you will halve the amount of time it takes.

Notice how scaling up or down involves multiplying or dividing values by a common factor.

A third use of the word 'proportion' occurs when people ask questions such as, 'What proportion of the tickets were sold to parents?' Two aspects of this use are important: firstly, the words 'proportion' and 'fraction' are interchangeable here; the question could equally well have been, 'What fraction of the tickets …'. And secondly, use of the word 'proportion' usually implies some kind of 'whole' that the proportion is 'of'; ratio comparisons don't imply any 'whole' value, but proportions do.

It is worth highlighting the fact that proportions are often described and expressed in terms of percentages. This is once again because we are essentially concerned with comparisons here, and percentages were designed to make comparisons easy. We would probably respond to the question above by saying, '70% of all the tickets were sold to parents'.

In Number, Pattern and Calculating 6, there is increased attention both to percentages and to ratio and proportion. The contexts in which this work is developed are crucial to children's understanding. Pie charts (see *Geometry, Measurement and Statistics 6*, Measurement 2) are particularly important illustrations in this context, and the scaling up or down of recipes and maps and models are vital experiences readily related to everyday situations.

Connecting all this together …

You will have noticed, in reading the above, just how much all of these ratio comparisons and relationships depend upon children's underlying skills with and understanding of **multiplying** and **dividing**. Since (almost) all dividing can be converted to **equivalent** multiplying, you may now appreciate why all of this work comes to be collected together under the single heading of **multiplicative thinking**.

Multiplying, dividing, ratios, proportion, fractions, decimals, and percentages all link intricately with each other to produce a very varied and complex set of ways to communicate about essentially the same kind of relationships – **multiplicative** relationships. These relationships, and the ways that we communicate about them mathematically, are crucial to children's future progress in both mathematics and in science, as well as in everyday life; they deserve our and our children's full attention. 'Converting' between multiplying and dividing, between common fractions and decimals, between fractions and percentages, between improper fractions and mixed numbers and so on is crucial to success.

Interestingly, in the 2014 National Curriculum,[1] a new term appeared with the aim of collecting together a whole range of problem situations that invite **multiplicative thinking**; the term introduced was 'correspondence problems', probably because this work will also underlie the significant later development of those other very important mathematical relationships – **functions** – and functions are essentially about how changes in one variable 'correspond' to changes in another.

In what follows we offer, and conclude with, some explanation and advice on how to connect a range of problems together that will invite children to **think multiplicatively**. When offering these types of problems to children it is important not to suggest or to tell them in advance whether they should multiply, or when to divide; their long-term success depends upon children thinking these things out for themselves.

1 See Year 4 in Mathematics programmes of study: key stages 1 and 2 National curriculum in England 2014.

Correspondence problems

Correspondence problems refer to a variety of situations in which there is said to be a one-to-many correspondence of some kind between two sets of objects. A simple example would be when a number of children are sharing a number of apples; the two sets are the children and the apples. Another example would be when someone has a number of hats and a number of coats and can combine these to make a range of different outfits. In this case, the set of hats 'corresponds' with the set of coats to produce outfits. A third example would be a correspondence between cars and their wheels; each car would correspond to four wheels (usually). Note that, although the correspondence is technically called 'one-to-many', in general the number of elements in each set in a situation can be anything. As a result, such situations are sometimes described as contexts in which n objects correspond to m objects. In practice, there can be the same number of elements in each set ($n = m$), and which set is specified first (the larger or the smaller) doesn't matter.

The point about correspondence problems is that they usually require **multiplicative thinking**. If one car has four wheels, how many wheels will six cars have? If there are three apples to share equally between twelve children, how much apple will each child get? If a person has five hats and three coats, how many different outfits can they put together? If I have 48 wheels available, how many cars could I make?

Note that, although the thinking involved in all these different situations is called 'multiplicative', some would lead us to multiply and some situations would probably lead us to divide two numbers.

Essentially, a one-to-many correspondence is a way of describing a **ratio**, which could also be expressed as $n:m$ or n to m. Such ratios are at the heart of multiplicative situations. In the sharing case of apples and children, the ratio of apples to children is 3 : 12, and each child gets 'three twelfths' of a cake or $\frac{3}{12}$ (written as a fraction).

When we come to hats and coats, there is no sharing going on, but a 'multiplying' of possibilities. For every one of three coats, we can put on one of five different hats, giving 1 × 5 possibilities for each coat (ratio 1 : 5); with three coats the possibilities become (1 × 5) + (1 × 5) + (1 × 5) = 3 × 5 = 15. With wheels and cars, the ratio is 4 : 1, so there will always be four times as many wheels as cars.

The thing to remember is that 'correspondence problems' are all essentially to do with '**ratio** situations', but that we have to think carefully each time whether to multiply or divide to answer a particular question. Later on, as children develop their **multiplicative thinking**, they will come to understand that all dividing can be done by multiplying with fractions, and that there's a reason why fractions are called rational numbers. Make sure that children experience a wide variety of correspondence problems and encourage them to think very carefully about what kind of an answer would make sense in each situation.